Outer Bounds for Gaussian Multiple Access Channels with State Known at One Encoder

Wei Yang[1], Yingbin Liang[2], Shlomo Shamai (Shitz)[3], and H. Vincent Poor[1]

[1]Princeton University, Princeton, NJ, 08544 USA
[2]Syracuse University, Syracuse, NY 13244 USA
[3]Technion–Israel Institute of Technology, Technion City, Haifa 32000, Israel

***Abstract*—This paper studies a two-user state-dependent Gaussian multiple-access channel with state noncausally known at one encoder. Two new outer bounds on the capacity region are derived, which improve uniformly over the best known (genie-aided) outer bound. The two corner points of the capacity region as well as the sum rate capacity are established, and it is shown that a single-letter solution is adequate to achieve both the corner points and the sum rate capacity. Furthermore, the full capacity region is characterized in situations in which the sum rate capacity is equal to the capacity of the helper problem. The proof exploits the optimal-transportation idea of Polyanskiy and Wu (which was used previously to establish an outer bound on the capacity region of the interference channel) and the worst-case Gaussian noise result for the case in which the input and the noise are dependent.**

I. Introduction

We study a two-user state-dependent Gaussian multiple-access channel (MAC) with state noncausally known at one encoder (see Fig. 1). The channel input-output relationship for a single channel use is given by

$$Y = X_1 + X_2 + S + Z \qquad (1)$$

where $Z \sim \mathcal{N}(0,1)$ denotes the additive white Gaussian noise, and X_1 and X_2 are the channel inputs from two users, which are subject to (average) power constraints P_1 and P_2, respectively. The state $S \sim \mathcal{N}(0,Q)$ is known noncausally at encoder 1 (state-cognitive user), but is not known at encoder 2 (non-cognitive user) nor at the decoder. This channel model generalizes Costa's dirty-paper channel [1] to the multiple-access setting, and is also known as "dirty MAC" or "MAC with a single dirty user" [2].

Although the capacity region of the dirty MAC described in (1) has been studied extensively in the literature [2]–[4], no single-letter expression for the capacity region is available to date. Kotagiri and Laneman [3] derived an inner bound on the capacity region using a generalized dirty paper coding scheme at the cognitive encoder, which allows arbitrary correlation between the input X_1 and the state S. Philosof *et al.* [2] showed that the same rate region can be achieved using lattice-based transmission. In general, it is not clear

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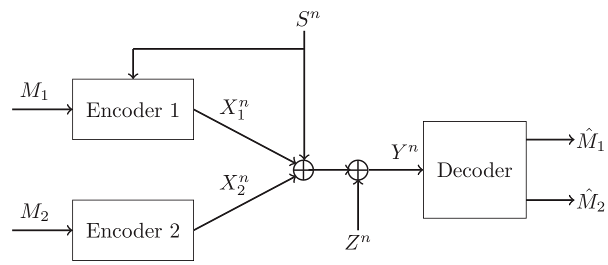

Figure 1. Gaussian MAC with additive Gaussian state available noncausally at one encoder.

whether a single-letter solution (i.e., random coding/random binning using independent and identically distributed (i.i.d.) copies of some scalar distribution) is optimal for the dirty MAC (1). However, as [2] and [4] demonstrated, a single-letter solution is suboptimal for the *doubly-dirty* MAC, in which the output is corrupted by two states, each known at one encoder noncausally. In this case, (linear) structured lattice coding outperforms the best known single-letter solution.

On the converse side, all existing outer bounds are obtained by assuming that a genie provides auxiliary information to the encoders/decoder. For example, by revealing the state to the decoder, one obtains an outer bound given by the capacity region of the Gaussian MAC without state dependence. Somekh-Baruch *et al.* [5] considered the setup in which the cognitive encoder knows the message of the non-cognitive encoder (also known as the dirty MAC with degraded message sets), and derived the exact capacity region. The resulting region is tighter than the trivial outer bound if the cognitive encoder's rate is above a threshold or if the state power Q is large. In [6], Zaidi *et al.* considered the case in which the non-cognitive encoder knows the message of the cognitive encoder (i.e., the roles of the two encoders are reversed), and derived another outer bound. To the best of our knowledge, no attempt has been made to outer-bound the capacity region of the dirty MAC (1) directly.

Different variants of the dirty MAC model in (1) have also been investigated in the literature. A special case of the dirty MAC model is the "helper problem" [7], in which the cognitive user does not send any information, and its goal is to help the non-cognitive user. For the helper problem, the capacity (of the non-cognitive user) is known for a wide range of channel parameters [8]. The authors in [9] and [10] considered the case in which the state is known only strictly causally or causally at the cognitive encoder(s), and derived inner and outer bounds on the capacity region. The capacity region of the MAC with action-dependent states was established in Dikstein *et al.* [11].

Contributions: The main contributions of this paper are the establishment of two new outer bounds on the capacity region of the dirty MAC (1). Differently from [5], [6], and [12], we do not assume degraded message sets or causal knowledge of the state at the non-cognitive encoder. Our bounds improve uniformly over the best known (genie-aided) outer bound (see Fig. 2 for a numerical example). Together with the generalized dirty paper coding inner bound in [3], the new outer bounds allow us to characterize the two corner points of the capacity region as well as the sum rate capacity (note that, unlike in [2], we do not assume $Q \to \infty$). This implies that a single-letter solution is adequate to achieve both the corner points and the sum rate capacity. Furthermore, the full capacity region is established in situations in which the sum rate capacity coincides with the capacity of the helper problem.

The proof of our outer bounds builds on a recent technique proposed by Polyanskiy and Wu [13] that bounds the difference of the differential entropies of two probability distributions via their quadratic Wasserstein distance and via Talagrand's transportation inequality [14]. It also relies on a generalized version of the worst-case Gaussian noise result, in which the Gaussian input and the noise are dependent [15]. We anticipate that these techniques can be useful more broadly for other state-dependent multiuser models, such as state-dependent interference channels and relay channels.

II. Problem Setup and Previous Results

Consider the Gaussian MAC (1) with additive Gaussian state noncausally known at encoder 1 depicted in Fig. 1. The state $S \sim \mathcal{N}(0, Q)$ is independent of the additive white Gaussian noise $Z \sim \mathcal{N}(0,1)$ and of the input X_2 of the non-cognitive encoder. The state and the noise are i.i.d. over channel uses. We assume that encoder 1 and encoder 2 must satisfy the (average) power constraints[1]

$$\sum_{i=1}^{n} \mathbb{E}\left[X_{1,i}^2(M_1, S^n)\right] \le nP_1 \tag{2}$$

$$\sum_{i=1}^{n} \mathbb{E}\left[X_{2,i}^2(M_2)\right] \le nP_2 \tag{3}$$

where the index i denotes the channel use, and M_1 and M_2 denote the transmitted messages, which are independently and uniformly distributed.

Let $\mathcal{C}(P_1, P_2, Q)$ be the capacity region of the dirty Gaussian MAC (1). A single-letter characterization for $\mathcal{C}(P_1, P_2, Q)$ is not known. The best known achievable rate region was derived by Kotagiri and Laneman [3], and is achieved by generalized dirty paper coding. The best known outer bound is given by the region of rate pairs (R_1, R_2) satisfying[2]

$$R_1 \le \frac{1}{2}\log(1 + P_1(1-\rho_1^2-\rho_s^2)) \tag{4}$$

$$R_2 \le \frac{1}{2}\log\left(1 + \frac{P_2(1-\rho_1^2-\rho_s^2)}{1-\rho_s^2}\right) \tag{5}$$

$$\begin{aligned} R_1 + R_2 \le{}& \frac{1}{2}\log(1 + P_1(1-\rho_1^2-\rho_s^2)) \\ &+ \frac{1}{2}\log\Big(1 + \frac{(\sqrt{P_2}+\rho_1\sqrt{P_1})^2}{1 + P_1(1-\rho_1^2-\rho_s^2) + (\sqrt{Q}+\rho_s\sqrt{P_1})^2}\Big) \end{aligned} \tag{6}$$

$$R_1 + R_2 \le \frac{1}{2}\log(1 + P_1 + P_2) \tag{7}$$

for some $\rho_1 \in [0,1]$ and $\rho_s \in [-1, 0]$ that satisfy $\rho_1^2 + \rho_s^2 \le 1$. This outer bound is a combination of several (genie-aided) outer bounds established in the literature:

- The bounds (4) and (6) characterize the capacity region of the dirty MAC under the additional assumption that the cognitive user knows the message of the non-cognitive user [5].
- The bounds (5) and (6) form an outer bound on the capacity region of the dirty MAC under the additional assumption that the non-cognitive user knows the message of the cognitive user [6].
- The bound (7) upper-bounds the sum rate of the Gaussian MAC without state dependence.

As reviewed in Section I, the dirty MAC model includes the helper problem as a special case. More specifically, in the helper problem, the cognitive user does not send any information, and its goal is to assist the non-cognitive user by canceling the state. The capacity of the helper problem is

$$C_{\text{helper}} \triangleq \max\{R_2 : (0, R_2) \in \mathcal{C}(P_1, P_2, Q)\}. \tag{8}$$

Sun *et al.* [8, Th. 2] recently proved that

$$C_{\text{helper}} = \frac{1}{2}\log(1 + P_2) \tag{9}$$

provided that P_1, P_2, and Q satisfy the following condition.

Condition 1: There exists an $\alpha \in [1-\sqrt{P_1/Q}, 1+\sqrt{P_1/Q}]$ such that

$$(P_1 - (\alpha-1)^2 Q)^2 \ge \alpha^2 Q(P_2 + 1 - P_1 + (\alpha-1)^2 Q). \tag{10}$$

In other words, if Condition 1 is satisfied, then the state can be perfectly canceled, and the non-cognitive user achieves the channel capacity without state dependence. Note that, to satisfy Condition 1 it is not necessary that $P_1 \ge Q$ (e.g., (10) holds as long as $P_1 \ge P_2 + 1$, regardless of the value of Q).

III. Main Results

A. New outer bounds

The main results of this paper are new outer bounds on the capacity region. For notational convenience, we denote

$$C_1 \triangleq \frac{1}{2}\log(1 + P_1), \quad C_2 \triangleq \frac{1}{2}\log(1 + P_2) \tag{11}$$

[1] Note that, the authors of [2] and [5] assumed *per-codeword* power constraints, i.e., for all messages m_1 and m_2, the codewords x_1^n and x_2^n satisfy $\sum_{i=1}^n x_{1,i}^2(m_1, S^n) \le nP_1$ and $\sum_{i=1}^n X_{2,i}^2(m_2) \le nP_2$ almost surely. Clearly, every outer bound for the average power constraint is also a valid outer bound for the *per-codeword* power constraint.

[2] In this paper, the logarithm (log) and exponential (exp) functions are taken with respect to an arbitrary basis.

which are the maximum achievable rates of user 1 and user 2, respectively. Due to space constraints, we have omitted the proofs of most results. They can be found in [16].

Theorem 1: The capacity region $\mathcal{C}(P_1, P_2, Q)$ of the dirty MAC (1) is outer-bounded by the region with rate pairs (R_1, R_2) satisfying

$$R_2 \leq C_{\mathrm{helper}} \tag{12}$$

and

$$R_1 \leq \min_{0\leq\delta\leq 1} \left\{\frac{1}{2}\log\left(1+\frac{1+P_2-\delta}{P_2\delta}g(R_2)\right)+f(\delta)\right\} \tag{13}$$

where

$$g(R_2) \triangleq \exp\left(2c_1\sqrt{C_2-R_2}+2(C_2-R_2)\right)-1 \tag{14}$$

with

$$c_1 \triangleq \frac{3\sqrt{1+(\sqrt{P_1}+\sqrt{Q})^2+P_2}+4(\sqrt{P_1}+\sqrt{Q})}{\sqrt{(1+P_2)/(2\log e)}} \tag{15}$$

and

$$f(\delta) \triangleq \max_{\rho\in[-1,0]} \frac{1}{2}\left\{\log\frac{1+P_2+P_1+Q+2\rho\sqrt{P_1Q}}{\delta+P_1+Q+2\rho\sqrt{P_1Q}} + \log\frac{\delta+(1-\rho^2)P_1}{1+P_2}\right\}. \tag{16}$$

Remark 1: The objective function on the right-hand side (RHS) of (16) is concave in ρ for every $\delta \in [0, 1]$.

The proof of Theorem 1 is sketched in Section IV.

The outer bound provided in Theorem 1 improves the best known outer bound in the regime where R_2 is close to C_2 (provided that C_{helper} is also close to C_2). The next theorem provides a tighter upper bound on the sum rate than (6) and (7).

Theorem 2: The capacity region $\mathcal{C}(P_1, P_2, Q)$ of the dirty MAC (1) is outer-bounded by the region with rate pairs (R_1, R_2) satisfying

$$R_1 \leq \frac{1}{2}\log(1+P_1(1-\rho^2)) \tag{17}$$

$$R_2 \leq C_{\mathrm{helper}} \tag{18}$$

$$R_1+R_2 \leq \frac{1}{2}\log\left(1+\frac{P_2}{1+P_1+Q+2\rho\sqrt{P_1Q}}\right) + \frac{1}{2}\log(1+P_1(1-\rho^2)) \tag{19}$$

for some $\rho \in [-1, 0]$.

B. Sum rate capacity

Let C_{sum} be the sum rate capacity of the dirty MAC (1), i.e.,

$$C_{\mathrm{sum}} \triangleq \max\{R_1+R_2 : (R_1, R_2) \in \mathcal{C}(P_1, P_2, Q)\}. \tag{20}$$

By comparing the inner bound in [3] and the outer bound (19), we establish the sum rate capacity C_{sum}.

Theorem 3: The sum rate capacity of the dirty MAC (1) is

$$C_{\mathrm{sum}} = \max_{\rho\in[-1,0]} \frac{1}{2}\left\{\log\left(1+\frac{P_2}{1+P_1+Q+2\rho\sqrt{P_1Q}}\right) + \frac{1}{2}\log(1+P_1(1-\rho^2))\right\} \tag{21}$$

or equivalently,

$$C_{\mathrm{sum}} = C_2 + f(1). \tag{22}$$

The next result establishes that, if $C_{\mathrm{helper}} = C_{\mathrm{sum}}$, then the outer bound in Theorem 2 matches the inner bound in [3]. In this case, we obtain a complete characterization of the capacity region $\mathcal{C}(P_1, P_2, Q)$.

Corollary 4: For the dirty MAC (1), if $C_{\mathrm{helper}} = C_{\mathrm{sum}}$, then the capacity region is the set of rate pairs

$$R_1 \leq \frac{1}{2}\log\big(1+P_1(1-\rho^2)\big) \tag{23}$$

$$R_1+R_2 \leq \frac{1}{2}\log\left(1+\frac{P_2}{1+P_1+Q+2\rho\sqrt{P_1Q}}\right) + \frac{1}{2}\log(1+P_1(1-\rho^2)) \tag{24}$$

for some $\rho \in [-1, 0]$.

C. Corner points

The bounds in Theorems 1 and 2 allow us to characterize the corner points of the capacity region, which are defined as

$$\widetilde{C}_1(P_1, P_2, Q) \triangleq \max\{R_1 : (R_1, C_2) \in \mathcal{C}(P_1, P_2, Q)\} \tag{25}$$

$$\widetilde{C}_2(P_1, P_2, Q) \triangleq \max\{R_2 : (C_1, R_2) \in \mathcal{C}(P_1, P_2, Q)\}. \tag{26}$$

Corollary 5: For every P_1, every P_2, and every Q, we have

$$\widetilde{C}_2(P_1, P_2, Q) = \frac{1}{2}\log\left(1+\frac{P_2}{1+P_1+Q}\right). \tag{27}$$

Furthermore, if P_1, P_2, and Q satisfy Condition 1, then

$$\widetilde{C}_1(P_1, P_2, Q) = f(0) \tag{28}$$

where $f(\cdot)$ is defined in (16).

Proof: The corner point (27) follows from (17) and (19) (with $\rho = 0$), and (28) follows from (13) by setting $R_2 = C_2$, and by taking $\delta \to 0$. ■

A few remarks are in order.

- The bottom corner point $(C_1, \widetilde{C}_2)$ also follows from the (genie-aided) outer bound in (4) and (6).
- The following three statements are equivalent [16]:
  1) $f(0) \geq 0$;
  2) $C_{\mathrm{helper}} = C_2$;
  3) The parameters P_1, P_2, Q satisfy Condition 1.
- In the asymptotic limit of strong state power (i.e., $Q \to \infty$), the two corner points become

$$\lim_{Q\to\infty} \widetilde{C}_1(P_1, P_2, Q) = \frac{1}{2}\log\frac{P_1}{1+P_2} \tag{29}$$

$$\lim_{Q\to\infty} \widetilde{C}_2(P_1, P_2, Q) = 0. \tag{30}$$

For comparison, existing outer bounds in [2] and [6] only yield the upper bound

$$\lim_{Q\to\infty} \widetilde{C}_1(P_1, P_2, Q) \leq \frac{1}{2}\log\frac{1+P_1}{1+P_2}. \tag{31}$$

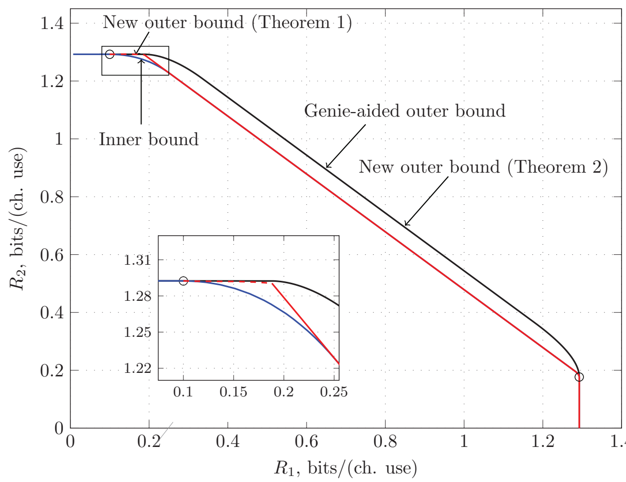

Figure 2. Inner and outer bounds on the capacity region region $\mathcal{C}(P_1, P_2, Q)$ with $P_1 = 5$, $P_2 = 5$, and $Q = 12$.

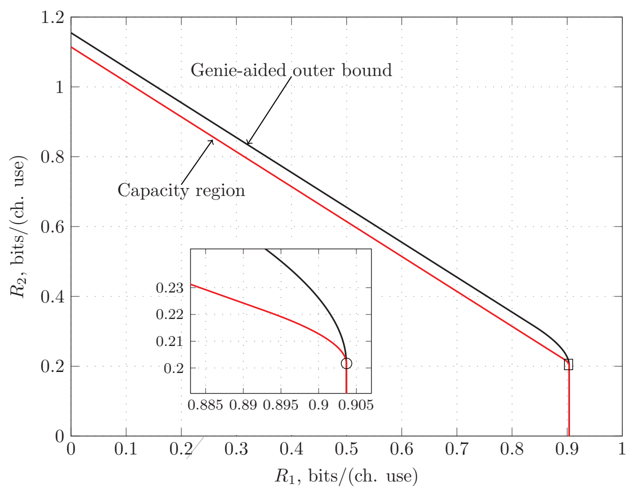

Figure 3. A comparison between the capacity region $\mathcal{C}(P_1, P_2, Q)$ and the genie-aided outer bound with $P_1 = 2.5$, $P_2 = 5$, and $Q = 12$.

D. Numerical results

In Fig. 2, we compare our new bounds in Theorems 1 and 2 with the inner and outer bounds reviewed in Section II for $P_1 = 5$, $P_2 = 5$, and $Q = 12$. It is not difficult to verify that this set of parameters satisfy Condition 1. We make the following observations from Fig. 2.

- The top corner point of the capacity region is given by the rate pair $(1.29, 0.1)$.
- The outer bound in Theorem 2 matches the inner bound when $R_1 \geq 0.25$ bits/ch. use.
- In the regime $R_1 \in (0.1, 0.25)$, there is a gap between our outer bounds and the inner bound. This regime can be further divided into two regimes: if $R_1 \in (0.1, 0.19)$, then Theorem 1 yields a tighter upper bound on R_2; if $R_1 \in (0.19, 0.25)$, then the bound in Theorem 2 is tighter.

Overall, our outer bounds provide a substantial improvement over the genie-aided outer bound in (4)–(7).

In Fig. 3, we consider another set of parameters with $P_1 = 2.5$, $P_2 = 5$, and $Q = 12$. In this case, we have $C_{\text{helper}} = C_{\text{sum}} = 1.11$ bits/ch. use, and the capacity region $\mathcal{C}(P_1, P_2, Q)$ is completely characterized by Corollary 4. We observe that the boundary of the capacity region consists of three pieces: a straight line connecting the two points $(0, C_{\text{helper}})$ and $(0.89, 0.22)$, a curved line connecting $(0.89, 0.22)$ and the bottom corner point $(0.9, 0.2)$, and a vertical line connecting the bottom corner point $(0.9, 0.2)$ and $(0.9, 0)$.

E. Generalization to MAC with non-Gaussian state

In the proofs of Theorems 1–3, the only place where we have used the Gaussianity of S^n is to optimize appropriate mutual information terms over $P_{X_1|S}$ (see, e.g., (46)). If the state sequence S^n is non-Gaussian (but is i.i.d.), then the upper bound (13) remains valid if $f(\delta)$ is replaced by

$$\tilde{f}(\delta) \triangleq \max_{P_{X_1|S}} \Big\{ I(X_1 + S; Y_G) - I(S; Y_\delta) \Big\}. \tag{32}$$

In this case, the top corner point becomes

$$\widetilde{C}_1 = \max_{P_{X_1|S}} \{ I(X_1 + S; Y_G) - I(X_1 + S; S) \} \tag{33}$$

and the sum rate capacity becomes

$$C_{\text{sum}} = \max_{P_{X_1|S} P_{X_2}} I(X_1; Y | X_2, S) + I(X_2; Y). \tag{34}$$

Furthermore, both (21) and (34) can be achieved by treating interference as noise for the non-cognitive user, and by using generalized dirty paper coding for the cognitive user.

IV. Sketch of the Proof of Theorem 1

The upper bound (12) is straightforward. The proof of (13) is sketched as follows. Let

$$R_1 \triangleq \frac{1}{n} I(M_1; Y^n), \quad R_2 \triangleq \frac{1}{n} I(X_2^n; Y^n). \tag{35}$$

As explained in [13], this definition of rate agrees with the operational definition asymptotically.

Consider two auxiliary channels

$$Y_G^n \triangleq X_1^n + S^n + G^n + Z^n \tag{36}$$
$$Y_\delta^n \triangleq X_1^n + S^n + \sqrt{\delta} Z^n \tag{37}$$

where $G^n \sim \mathcal{N}(0, P_2 \mathsf{I}_n)$ is a Gaussian vector having the same power as X_2^n, and $\delta \in (0, 1)$ is a constant. In words, Y_G^n is obtained from Y^n by replacing the codeword X_2^n with Gaussian interference of the same power, and Y_δ^n is obtained from Y^n by removing the interference X_2^n and by increasing the signal-to-noise ratio (SNR). By standard manipulations of mutual information terms, we obtain

$$nR_1 \leq \underbrace{I(X_1^n + S^n; Y_\delta^n) - I(X_1^n + S^n; Y_G^n)}_{\triangleq I_\delta} + \underbrace{I(X_1^n + S^n; Y_G^n) - I(S^n; Y_\delta^n)}_{J_\delta} + o(n). \tag{38}$$

We next bound the two terms I_δ and J_δ on the RHS of (38) separately. To upper-bound I_δ, we invoke an elegant argument of Polyanskiy and Wu [13], used in the derivation of the outer bound on the capacity region of Gaussian interference channels. More specifically, by repeating the steps in [13, Eqs. (41)–(43)], we obtain

$$D(P_{X_2^n+Z^n}\|P_{G^n+Z^n}) \le n(C_2 - R_2) \tag{39}$$

and

$$nR_2 = h(Y^n) - h(Y_G^n) + \frac{n}{2}\log\frac{N_S(1+P_2)}{N_S(1)} \tag{40}$$

where $D(\cdot\|\cdot)$ denotes the relative entropy between two probability distributions, and

$$N_S(\gamma) \triangleq \exp\left\{\frac{2}{n}h(X_1^n + S^n + \sqrt{\gamma}Z^n)\right\}. \tag{41}$$

Since $\mathbb{E}\left[\|X_1^n + S^n\|^2\right] \le n(\sqrt{P_1} + \sqrt{Q})^2$, by [13, Prop. 1], Talagrand's inequality [14], and (39), we can bound the entropy difference between Y^n and Y_G^n as

$$h(Y^n) - h(Y_G^n) \le nc_1\sqrt{C_2 - R_2} \tag{42}$$

where c_1 is defined in (15). Using (42) in (40), we obtain

$$\frac{N_S(1)}{N_S(1+P_2)} \le \frac{\exp\left(2c_1\sqrt{C_2-R_2} + 2(C_2 - R_2)\right)}{1+P_2}. \tag{43}$$

Using the concavity of $N_S(\gamma)$ (which follows from Costa's entropy power inequality [17]), we conclude that

$$I_\delta = \frac{n}{2}\log\frac{N_S(\delta)}{N_S(1+P_2)} + \frac{n}{2}\log\frac{1+P_2}{\delta} \tag{44}$$

$$\le \frac{n}{2}\log\left(1 + \frac{1+P_2-\delta}{P_2\delta}g(R_2)\right) \tag{45}$$

where $g(R_2)$ is defined in (14).

To upper-bound J_δ, we observe that the sequence S^n is i.i.d., that $P_{Y_G^n|X_1^n+S^n}$ is memoryless, and that the functional $P_{X_1|S} \mapsto I(X_1+S;Y_G) - I(S;Y_\delta)$ is concave (which follows since, for a fixed channel, mutual information is concave in the input distribution, and for a fixed input distribution, mutual information is convex in the channel). Here, Y_G and Y_δ are single-letter versions of Y_G^n and Y_δ^n, respectively. These observations allow us to single-letterize J_δ as

$$J_\delta \le n \max_{P_{X_1|S}:\mathbb{E}[X_1^2]\le P_1}\left\{I(X_1+S;Y_G) - I(S;Y_\delta)\right\}. \tag{46}$$

By the Gaussian saddle point property (namely, the Gaussian distribution is the best input distribution for a Gaussian noise, and is the worst noise distribution for a Gaussian input), it is natural to expect that the RHS of (46) is maximized when X_1 and S are jointly Gaussian. A rigorous proof of this result relies on the following lemma, which generalizes the well-known worst-case Gaussian noise result [18], [19] to the case in which the noise and the Gaussian input are dependent.

Lemma 6 ([15, Th. 1]): Suppose $\boldsymbol{X}_G \sim \mathcal{N}(\mathbf{0}, \mathsf{K}_X)$ and $\boldsymbol{Z}_G \sim \mathcal{N}(\mathbf{0}, \mathsf{K}_Z)$ are Gaussian random vectors in $\mathbb{R}^d$, and $\boldsymbol{Z}$ is a random vector in $\mathbb{R}^d$ with the same covariance matrix as $\boldsymbol{Z}_G$. Assume that $\boldsymbol{X}_G$ is independent of $\boldsymbol{Z}_G$, and that

$$\mathbb{E}\left[\boldsymbol{X}_G\boldsymbol{Z}^{\mathrm{T}}\right] = \mathbf{0}_{d\times d} \tag{47}$$

where the superscript $(\cdot)^{\mathrm{T}}$ denotes transposition. Then

$$I(\boldsymbol{X}_G;\boldsymbol{X}_G+\boldsymbol{Z}_G) \le I(\boldsymbol{X}_G;\boldsymbol{X}_G+\boldsymbol{Z}). \tag{48}$$

Using Lemma 6 and following algebraic manipulations, we obtain

$$J_\delta \le nf(\delta). \tag{49}$$

The proof is concluded by substituting (45) and (49) into (38).

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