A View of Information-Estimation Relations in Gaussian Networks

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Outline

- Review of Information-Estimation Relations for Gaussian Channel
- Insights on the Mutual Information and MMSE of Gaussian Broadcast and Gaussian Wiretap channels
- Generalization of the Ozarow-Wyner lower bound on the mutual information for discrete inputs on Gaussian noise channels
- Properties of the Generalized Gaussian distribution

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Notation

The p-norm of a random vector $\mathbf{U} \in \mathbb{R}^n$ is given by

$$\|\mathbf{U}\|_p^p = \frac{1}{n}\mathrm{E}\left[\mathrm{Tr}^{\frac{p}{2}}\left(\mathbf{U}\mathbf{U}^T\right)\right].$$

For $n = 1$, $\|U\|_p^p = \mathrm{E}[|U|^p]$.

Example: $\mathbf{U} \sim \mathcal{N}(\mathbf{0}, \mathbf{I})$

$$n\|\mathbf{U}\|_p^p = 2^{\frac{p}{2}}\frac{\Gamma\left(\frac{n+p}{2}\right)}{\Gamma\left(\frac{n}{2}\right)},$$

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt.$$

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Notation

Channel under consideration

$$\mathbf{Y}_{\mathrm{snr}} = \sqrt{\mathrm{snr}}\mathbf{X} + \mathbf{Z},$$

where $\mathbf{X}, \mathbf{Z} \in \mathbb{R}^n$ are independent.

Mutual Information

$$I_n(\mathbf{X}, \mathrm{snr}) = \frac{1}{n} I(\mathbf{X}; \mathbf{Y}_{\mathrm{snr}}),$$

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MMPE

We define the Minimum Mean p-th Error (MMPE) of estimating $\mathbf{X}$ from $\mathbf{Y}$ as

$$\mathrm{mmpe}(\mathbf{X}|\mathbf{Y};p) := \inf_f \|\mathbf{X} - f(\mathbf{Y})\|_p^p$$

We denote the optimal estimator as $f_p(\mathbf{X}|\mathbf{Y})$.
For $n = 1$:

$$\mathrm{mmpe}(X|Y;p) = \inf_f \mathrm{E}\left[|X - f(Y)|^p\right].$$

Example: for $p = 2$

$$\mathrm{mmpe}(\mathbf{X}|\mathbf{Y};p=2) = \mathrm{mmse}(\mathbf{X}|\mathbf{Y}),$$
$$f_{p=2}(\mathbf{X}|\mathbf{Y}) = \mathrm{E}[\mathbf{X}|\mathbf{Y}].$$

We shall denote

$$\mathrm{mmpe}(\mathbf{X}|\mathbf{Y};p) = \mathrm{mmpe}(\mathbf{X}, \mathrm{snr}, p).$$

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MMSE is Special

$$\mathrm{mmse}(\mathbf{X}, \mathrm{snr}) = \|\mathbf{X} - \mathbb{E}[\mathbf{X} \mid \mathbf{Y}]\|_2^2$$

The only Hilbert space norm

I-MMSE relationship

$$\frac{d}{d\mathrm{snr}} I_n(\mathbf{X}, \mathrm{snr}) = \frac{1}{2}\mathrm{mmse}(\mathbf{X}, \mathrm{snr})$$

$$I_n(\mathbf{X}, \mathrm{snr}) = \frac{1}{2}\int_0^{\mathrm{snr}} \mathrm{mmse}(\mathbf{X}, t)dt$$

D. Guo, S. Shamai, and S. Verdu ́, “Mutual information and minimum mean-square error in Gaussian channels,” *IEEE Trans. Inf. Theory*, vol. 51, no. 4, pp. 1261–1282, April 2005.

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More On The I-MMSE

Conditional Version

For a joint distribution over $(\mathbf{X}, \mathbf{U})$

$$I_n(\mathbf{X}; \sqrt{\text{snr}}\mathbf{X} + \mathbf{Z}|\mathbf{U}) = \frac{1}{2}\int_0^{\text{snr}} \text{mmse}(\mathbf{X}; \gamma|\mathbf{U}) d\gamma$$

Limiting Case

$$\lim_{n\to\infty} I_n(\mathbf{X}; \sqrt{\text{snr}}\mathbf{X} + \mathbf{Z}) = \lim_{n\to\infty} \frac{1}{2}\int_0^{\text{snr}} \text{mmse}(\mathbf{X}; \gamma) d\gamma$$
$$= \frac{1}{2}\int_0^{\text{snr}} \lim_{n\to\infty} \text{mmse}(\mathbf{X}; \gamma) d\gamma$$

By Dominated Convergence Theorem (DCT): Requires that the MMSE converges point wise and is bounded

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More On The I-MMSE

Conditional Version

For a joint distribution over $(\mathbf{X}, \mathbf{U})$

$$I_n(\mathbf{X}; \sqrt{\mathrm{snr}}\mathbf{X} + \mathbf{Z}|\mathbf{U}) = \frac{1}{2}\int_0^{\mathrm{snr}} \mathrm{mmse}(\mathbf{X}; \gamma|\mathbf{U}) d\gamma$$

Limiting Case

$$\begin{aligned}\lim_{n\to\infty} I_n(\mathbf{X}; \sqrt{\mathrm{snr}}\mathbf{X} + \mathbf{Z}) &= \lim_{n\to\infty} \frac{1}{2}\int_0^{\mathrm{snr}} \mathrm{mmse}(\mathbf{X}; \gamma) d\gamma \\ &= \frac{1}{2}\int_0^{\mathrm{snr}} \lim_{n\to\infty} \mathrm{mmse}(\mathbf{X}; \gamma) d\gamma\end{aligned}$$

Limiting Case with Fatou Lemma

$$\limsup_{n\to\infty} I_n(\mathbf{X}; \sqrt{\mathrm{snr}}\mathbf{X} + \mathbf{Z}) \leq \frac{1}{2}\int_0^{\mathrm{snr}} \limsup_{n\to\infty} \mathrm{mmse}(\mathbf{X}; \gamma) d\gamma$$

Fatou Lemma: Requires that the MMSE is bounded

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Bounds

Simple Bounds

Theorem: For any $\mathbf{X}$

$$\mathrm{mmse}(\mathbf{X}, \mathrm{snr}) \leq \frac{1}{\mathrm{snr}}.$$

For any $\mathbf{X}$ such that $\|\mathbf{X}\|_2 \leq 1$

$$\mathrm{mmse}(\mathbf{X}, \mathrm{snr}) \leq \frac{1}{1 + \mathrm{snr}},$$ LMMSE bound

the above inequality is achieved iff $\mathbf{X} \sim \mathcal{N}(0, \mathbf{I})$.

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Bounds on the MMPE

$$\mathrm{mmse}(\mathbf{X}, \mathrm{snr}) \le \min\left(\frac{1}{\mathrm{snr}}, \|\mathbf{X}\|_2^2\right)$$

Theorem. For $\mathrm{snr} \ge 0$, $0 < q \le p$, and input $\mathbf{X}$ and

$$\mathbf{Y} = \sqrt{\mathrm{snr}}\mathbf{X} + \mathbf{Z},$$

we have that

$$\mathrm{mmpe}(\mathbf{X}, \mathrm{snr}, p) \le \min\left(\frac{\|\mathbf{Z}\|_p^p}{\mathrm{snr}^{\frac{p}{2}}}, \|\mathbf{X}\|_p^p\right),$$

$$n^{\frac{p}{q}-1}\mathrm{mmpe}^{\frac{p}{q}}(\mathbf{X}, \mathrm{snr}, q) \le \mathrm{mmpe}(\mathbf{X}, \mathrm{snr}, p).$$

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MMPE with Gaussian Input

(Estimation of Gaussian Input.) For $\mathbf{X}_{\mathrm{G}} \sim \mathcal{N}(0, \sigma^2 \mathbf{I})$ and $p \geq 1$

$$\mathrm{mmpe}(\mathbf{X}_{\mathrm{G}}, \mathrm{snr}, p) = \frac{\sigma^p \|\mathbf{Z}\|_p^p}{(1+\mathrm{snr}\sigma^2)^{\frac{p}{2}}},$$

with the optimal estimator given by

$$f_p(\mathbf{X}_{\mathrm{G}} | \mathbf{Y} = \mathbf{y}) = \frac{\sigma^2 \sqrt{\mathrm{snr}}}{1+\mathrm{snr}\sigma^2} \mathbf{y}.$$

(Asymptotic Optimality of Gaussian Input.) For every $p \geq 1$, and a random variable $\mathbf{X}$ such that $\|\mathbf{X}\|_p^p \leq \sigma^p \|\mathbf{Z}\|_p^p$, we have

$$\mathrm{mmpe}(\mathbf{X}, \mathrm{snr}, p) \leq k_{p,\sigma^2 \mathrm{snr}} \cdot \frac{\sigma^p \|\mathbf{Z}\|_p^p}{(1+\mathrm{snr}\sigma^2)^{\frac{p}{2}}}.$$

where

$$\begin{cases} k_{p,\sigma^2 \mathrm{snr}} = 1 & p = 2, \\ 1 \leq k_{p,\mathrm{snr}}^{\frac{1}{p}} = \frac{1+\sqrt{\mathrm{snr}}}{\sqrt{1+\mathrm{snr}}} \leq 1 + \frac{1}{\sqrt{1+\mathrm{snr}}} & p \neq 2. \end{cases}$$

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MMSE Bounds: Single Crossing Point Property

(**SCPP.**) For any fixed $\mathbf{X}$, suppose that $\text{mmse}(\mathbf{X}, \text{snr}_0) = \frac{\beta}{1+\beta\text{snr}_0}$, for some fixed $\beta \geq 0$. Then

$$\text{mmse}(\mathbf{X}, \text{snr}) \leq \frac{\beta}{1+\beta\text{snr}}, \quad \text{snr} \in [\text{snr}_0, \infty)$$

$$\text{mmse}(\mathbf{X}, \text{snr}) \geq \frac{\beta}{1+\beta\text{snr}}, \quad \text{snr} \in [0, \text{snr}_0).$$

SCPP+I-MMSE important tool for derive converses:

- Version of EPI
- BC
- Wiretap
- MIMO channels
- Bottleneck problems

D. Guo, Y. Wu, S. Shamai, and S. Verdu, “Estimation in Gaussian noise: Properties of the minimum mean-square error,” IEEE Trans. Inf. Theory, vol. 57, no. 4, pp. 2371–2385, April 2011.

R. Bustin, R. Schaefer, H. Poor, and S. Shamai, “On MMSE properties of optimal codes for the Gaussian wiretap channel,” in Proc. IEEE Inf. Theory Workshop, April 2015, pp. 1–5.

R. Bustin, M. Payaro, D. Palomar, and S. Shamai, “On MMSE crossing properties and implications in parallel vector Gaussian channels,” IEEE Trans. Inf. Theory, vol. 59, no. 2, pp. 818–844, Feb 2013.

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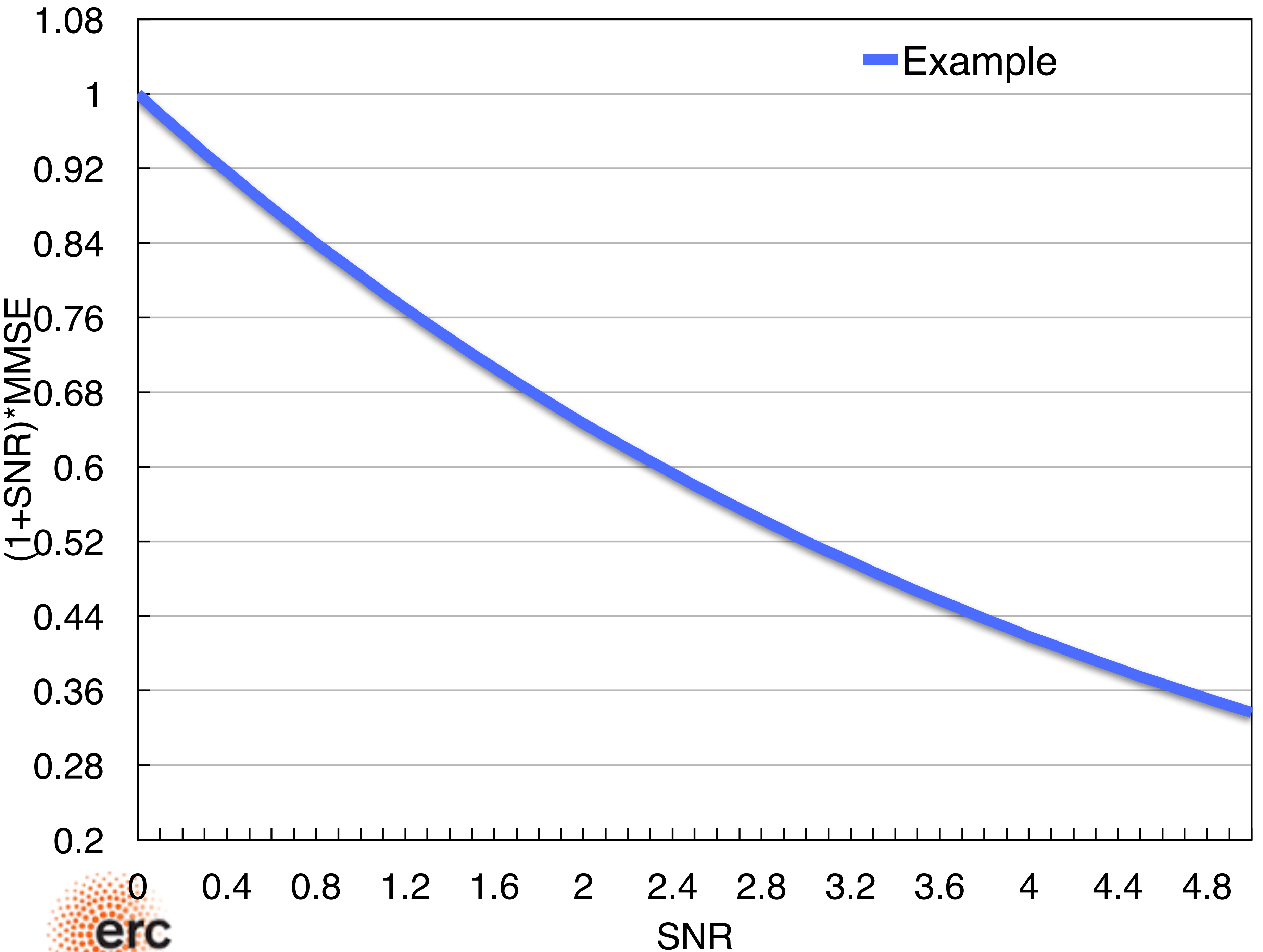
Review: MMSE bounds



$X_D = \{1, -1\}$

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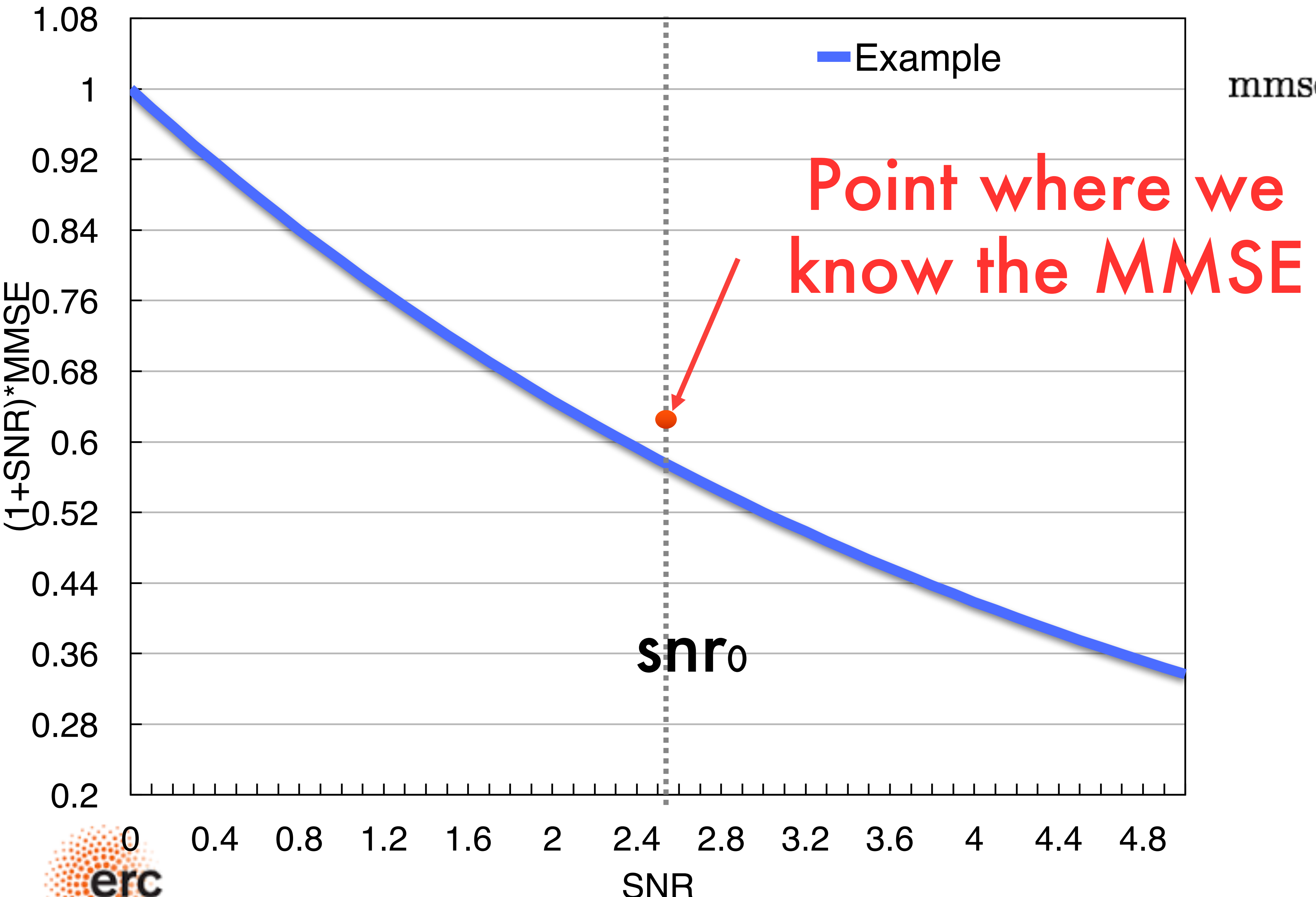
Review: MMSE bounds



$$\mathrm{mmse}(\mathbf{X}, \mathrm{snr}_0) = \frac{\beta}{1 + \beta \mathrm{snr}_0}$$

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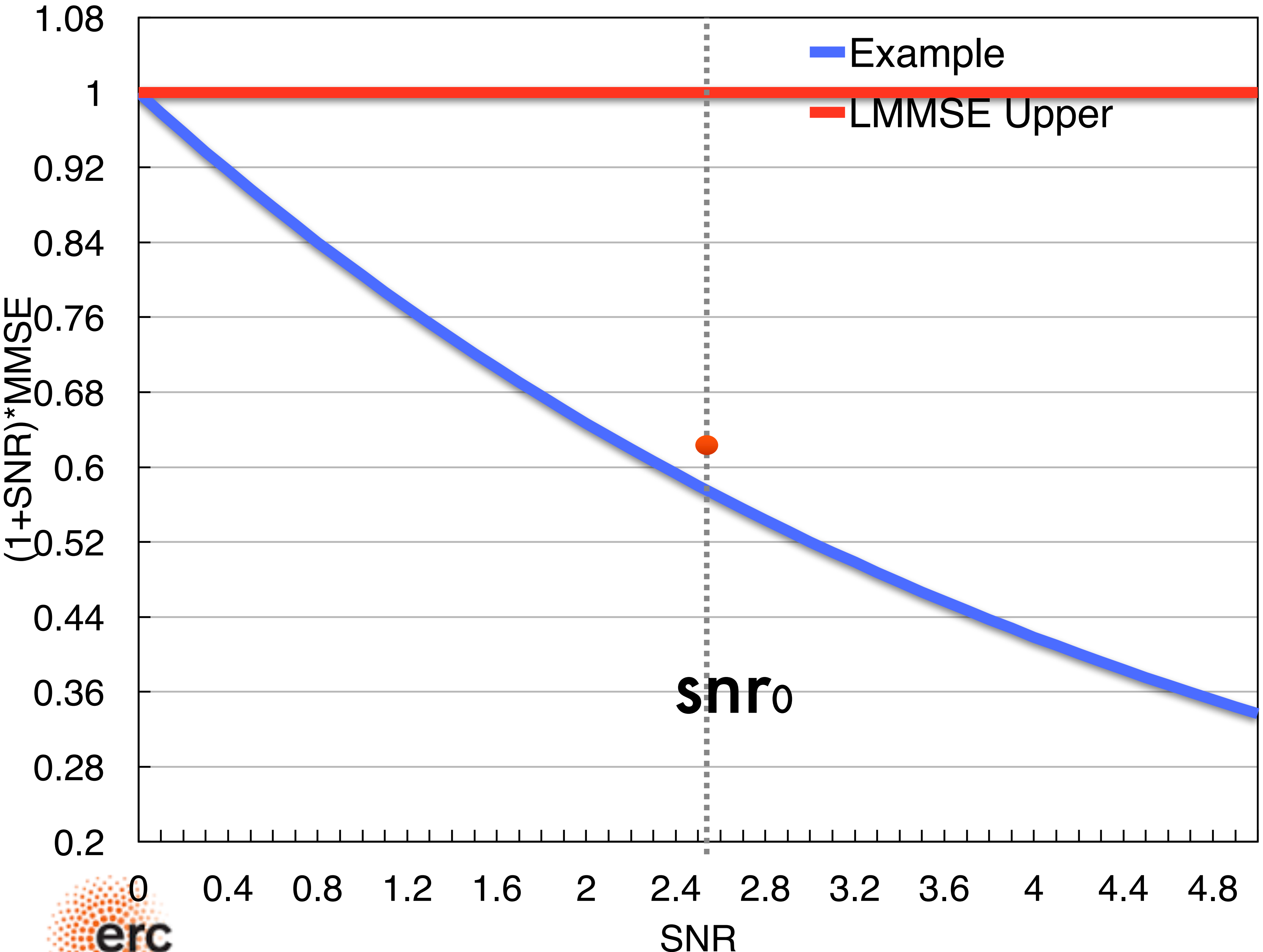
Review: MMSE bounds



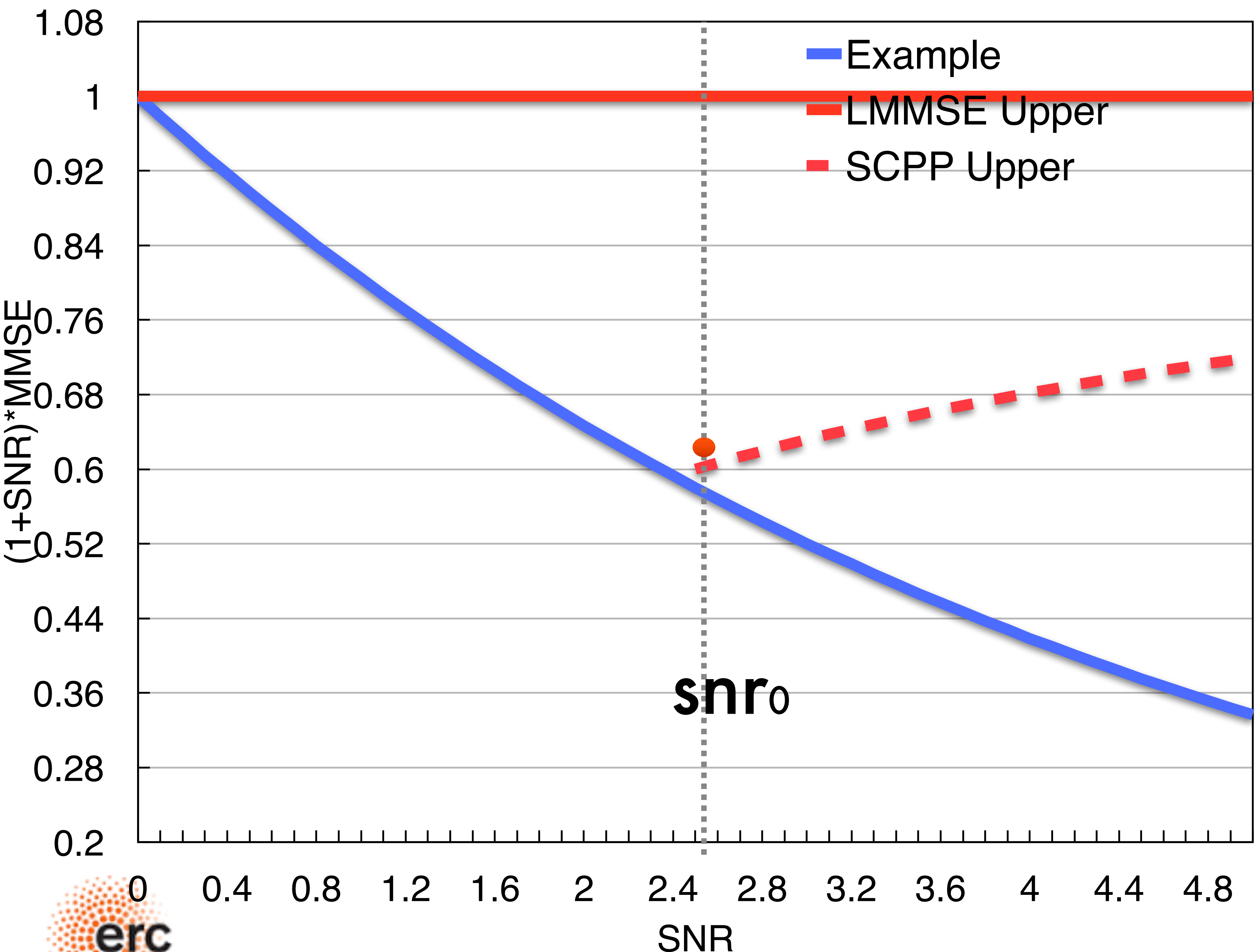
$$\mathrm{mmse}(\mathbf{X}, \mathrm{snr}_0) = \frac{\beta}{1 + \beta \mathrm{snr}_0}$$

LMMSE u.b.

$$\mathrm{mmse}(\mathbf{X}, \mathrm{snr}) \leq \frac{1}{1 + \mathrm{snr}}$$

only power constraint

Review: MMSE bounds



$$\mathrm{mmse}(\mathbf{X}, \mathrm{snr}_0) = \frac{\beta}{1+\beta \mathrm{snr}_0}$$

LMMSE u.b.

$$\mathrm{mmse}(\mathbf{X}, \mathrm{snr}) \leq \frac{1}{1+\mathrm{snr}}$$

SCPP u.b.

$$\mathrm{mmse}(\mathbf{X}, \mathrm{snr}) \leq \frac{\beta}{1+\beta \mathrm{snr}}$$

$$\text{for } \mathrm{snr} \geq \mathrm{snr}_0$$

MMSE constraint at snr_0

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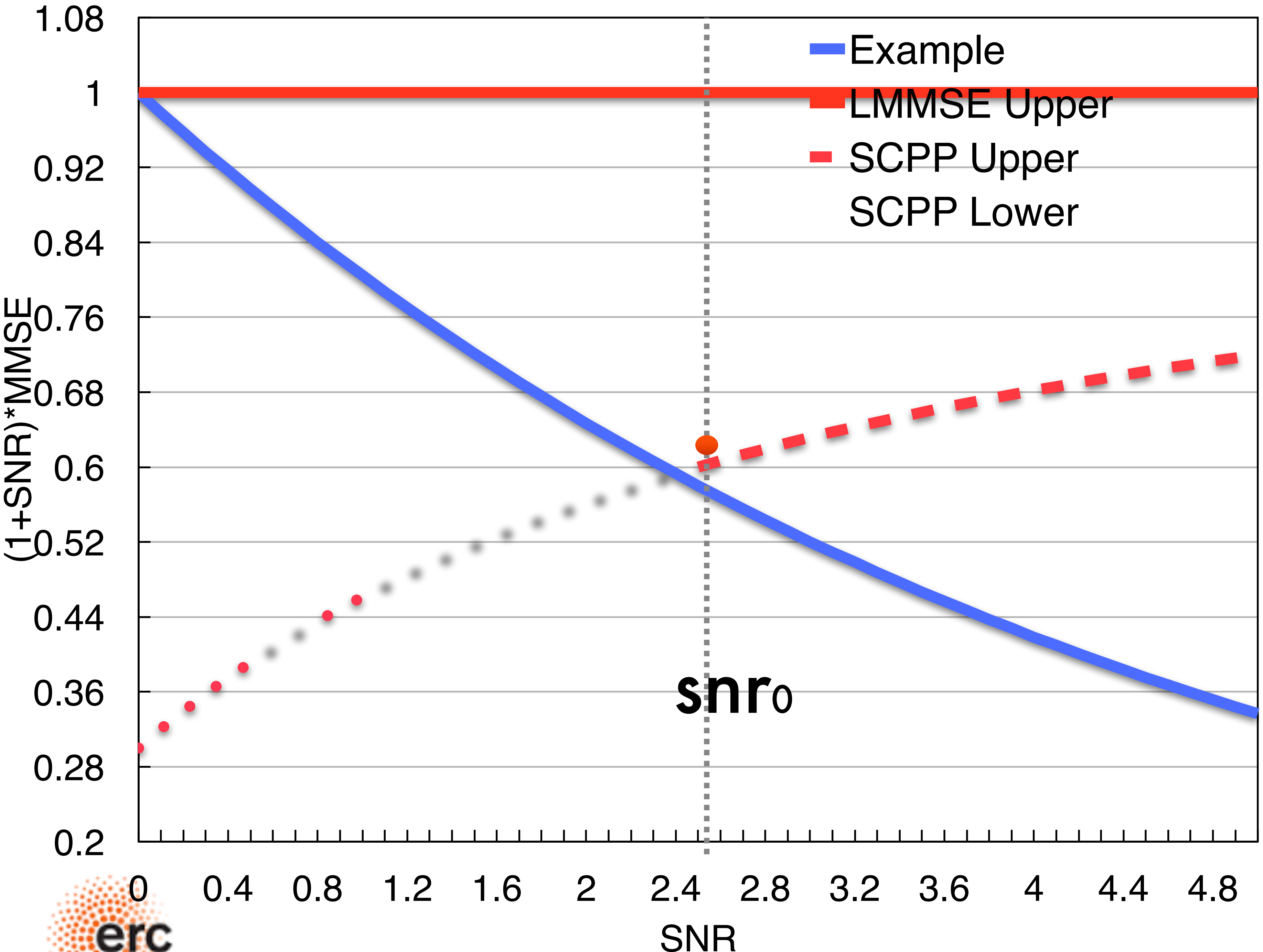
Review: MMSE bounds



$$\mathrm{mmse}(\mathbf{X}, \mathrm{snr}_0) = \frac{\beta}{1 + \beta \mathrm{snr}_0}$$

LMMSE u.b.

$$\mathrm{mmse}(\mathbf{X}, \mathrm{snr}) \leq \frac{1}{1 + \mathrm{snr}}$$

SCPP u.b.

$$\mathrm{mmse}(\mathbf{X}, \mathrm{snr}) \leq \frac{\beta}{1 + \beta \mathrm{snr}}$$

for $\mathrm{snr} \geq \mathrm{snr}_0$

SCPP l.b.

$$\mathrm{mmse}(\mathbf{X}, \mathrm{snr}) \geq \frac{\beta}{1 + \beta \mathrm{snr}}$$

for $\mathrm{snr} \leq \mathrm{snr}_0$

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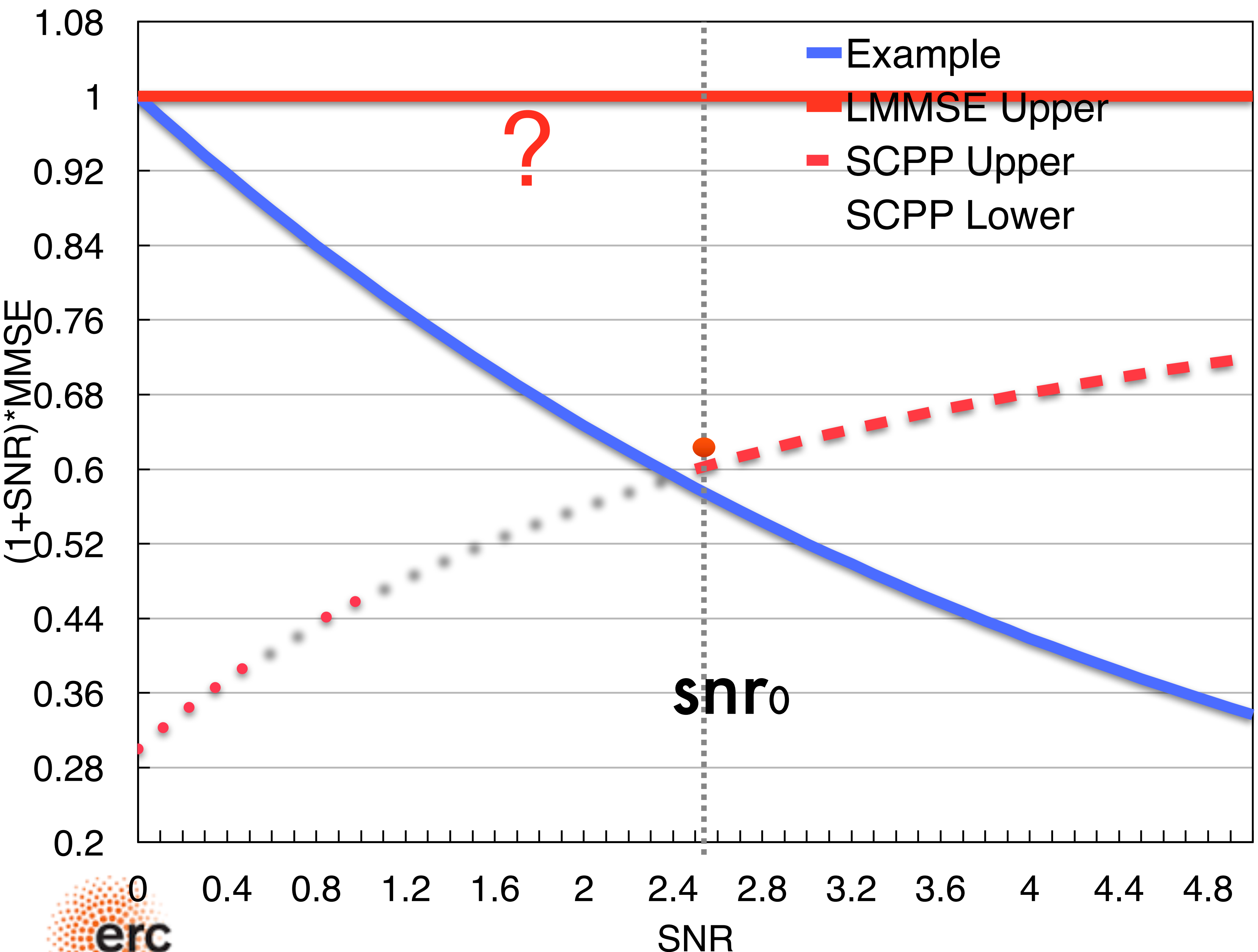
Review: MMSE bounds



$$\mathrm{mmse}(\mathbf{X}, \mathrm{snr}_0) = \frac{\beta}{1 + \beta \mathrm{snr}_0}$$

LMMSE u.b.

$$\mathrm{mmse}(\mathbf{X}, \mathrm{snr}) \leq \frac{1}{1 + \mathrm{snr}}$$

SCPP u.b.

$$\mathrm{mmse}(\mathbf{X}, \mathrm{snr}) \leq \frac{\beta}{1 + \beta \mathrm{snr}}$$

for $\mathrm{snr} \geq \mathrm{snr}_0$

SCPP l.b.

$$\mathrm{mmse}(\mathbf{X}, \mathrm{snr}) \geq \frac{\beta}{1 + \beta \mathrm{snr}}$$

for $\mathrm{snr} \leq \mathrm{snr}_0$

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max-MMSE problem

Motivated by the need of a complementary (to the SCPP) upper bound for MMSE

max-MMSE problem

$$\mathrm{M}_n(\mathrm{snr}, \mathrm{snr}_0, \beta) := \sup_{\mathbf{X}} \mathrm{mmse}(\mathbf{X}, \mathrm{snr}),$$

$$\text{s.t. } \|\mathbf{X}\|_2^2 \le 1,$$

$$\text{and } \mathrm{mmse}(\mathbf{X}, \mathrm{snr}_0) \le \frac{\beta}{1+\beta\mathrm{snr}_0}.$$

In the rest focus on $\mathrm{snr} < \mathrm{snr}_0$

From SCPP

$$\mathrm{mmse}(\mathbf{X}, \mathrm{snr}) \le \frac{\beta}{1+\beta\mathrm{snr}}$$

$$\text{for } \mathrm{snr} \ge \mathrm{snr}_0$$

solution for $\mathrm{snr} > \mathrm{snr}_0$: Gaussian input with reduced power

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Bounds on max-MMSE

$$\mathrm{M}_\infty(\mathrm{snr}, \mathrm{snr}_0, \beta) = \begin{cases} \frac{1}{1+\mathrm{snr}}, & \mathrm{snr} < \mathrm{snr}_0, \\ \frac{\beta}{1+\beta\mathrm{snr}}, & \mathrm{snr} \geq \mathrm{snr}_0, \end{cases} .$$

- Converse: LMMSE+SCPP.
- Achievability: superposition coding with Gaussian codebooks.

Discontinuity, or *phase transition,* at snr=snr_0

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N. Merhav, D. Guo, and S. Shamai, "Statistical physics of signal estimation in Gaussian noise: Theory and examples of phase transitions," *IEEE Trans. Inf. Theory*, vol. 56, no. 3, pp. 1400–1416, March 2010.

Bounds on max-MMSE

Theorem (Bound 1): For $\mathrm{snr} \le \mathrm{snr}_0$

$$\mathrm{mmse}(\mathbf{X}, \mathrm{snr}) \le \mathrm{mmse}(\mathbf{X}, \mathrm{snr}_0) + (n+2)\left(\frac{1}{\mathrm{snr}} - \frac{1}{\mathrm{snr}_0}\right).$$

A. Dytso, R. Bustin, D. Tuninetti, N. Devroye, S. Shamai, and H. V. Poor, “New bounds on MMSE and applications to communication with the disturbance constraint,” Submitted to *IEEE Trans. Inf. Theory,* https://arxiv.org/pdf/1603.07628, 2016.

Theorem (Bound 2): For $0 < \mathrm{snr} \le \mathrm{snr}_0$,

$$\mathrm{mmse}(\mathbf{X}, \mathrm{snr}) \le \min_{r > \frac{2}{\gamma}} \kappa(r,\gamma,n)\left(\mathrm{mmse}(\mathbf{X}, \mathrm{snr}_0)\right)^{\frac{\gamma r - 2}{r-2}},$$

where $\gamma := \frac{\mathrm{snr}}{2\mathrm{snr}_0 - \mathrm{snr}} \in (0,1]$, and

$$\kappa(r,\gamma,n) := \frac{\sqrt{2}}{n^{1-\gamma}}\left(\frac{1+\gamma}{\gamma}\right)^{\frac{n(1-\gamma)-1}{2}} M_r^{\frac{2(1-\gamma)}{r-2}}, \quad M_r = 2^r \min\left(\frac{\|\mathbf{Z}\|_r^r}{\mathrm{snr}_0^{\frac{r}{2}}}, \|\mathbf{X}\|_r^r\right).$$

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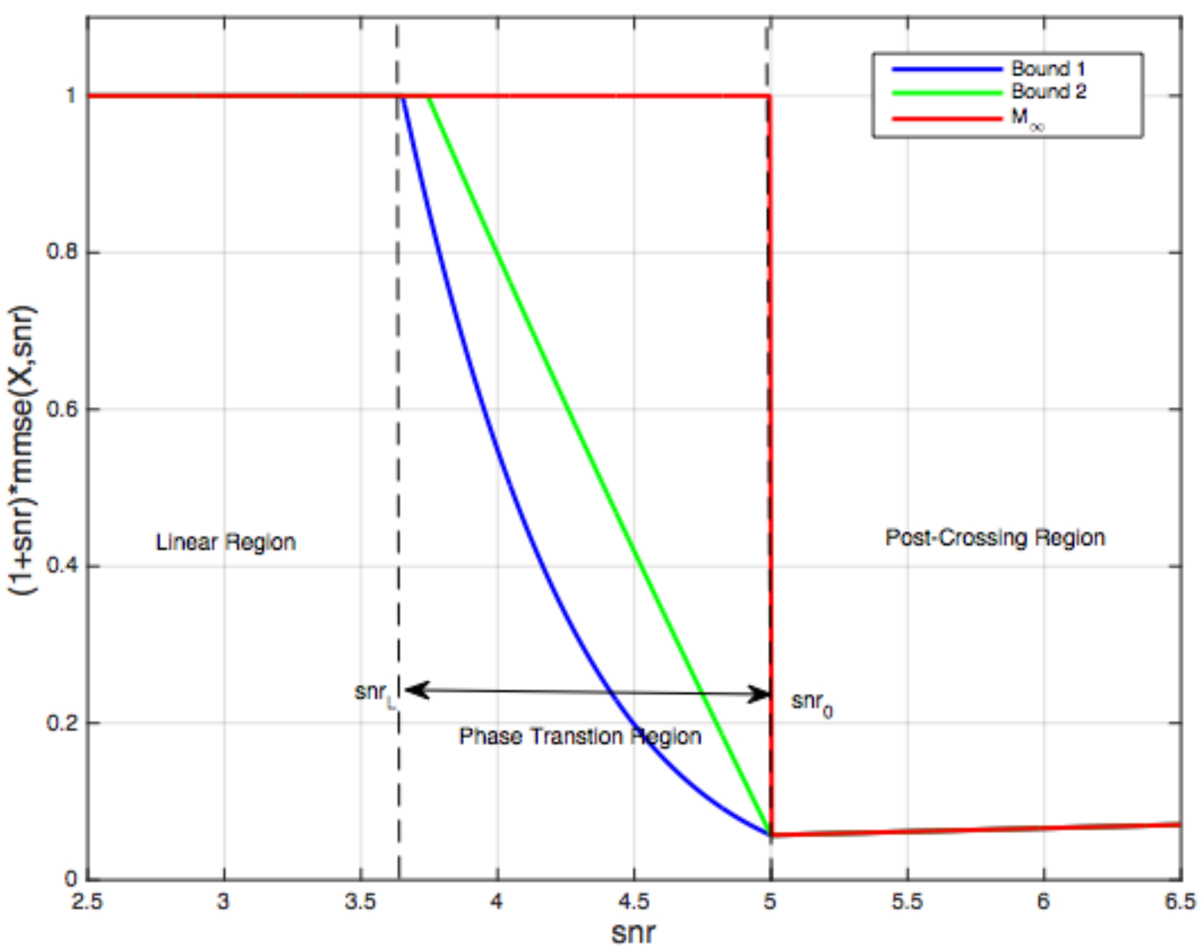
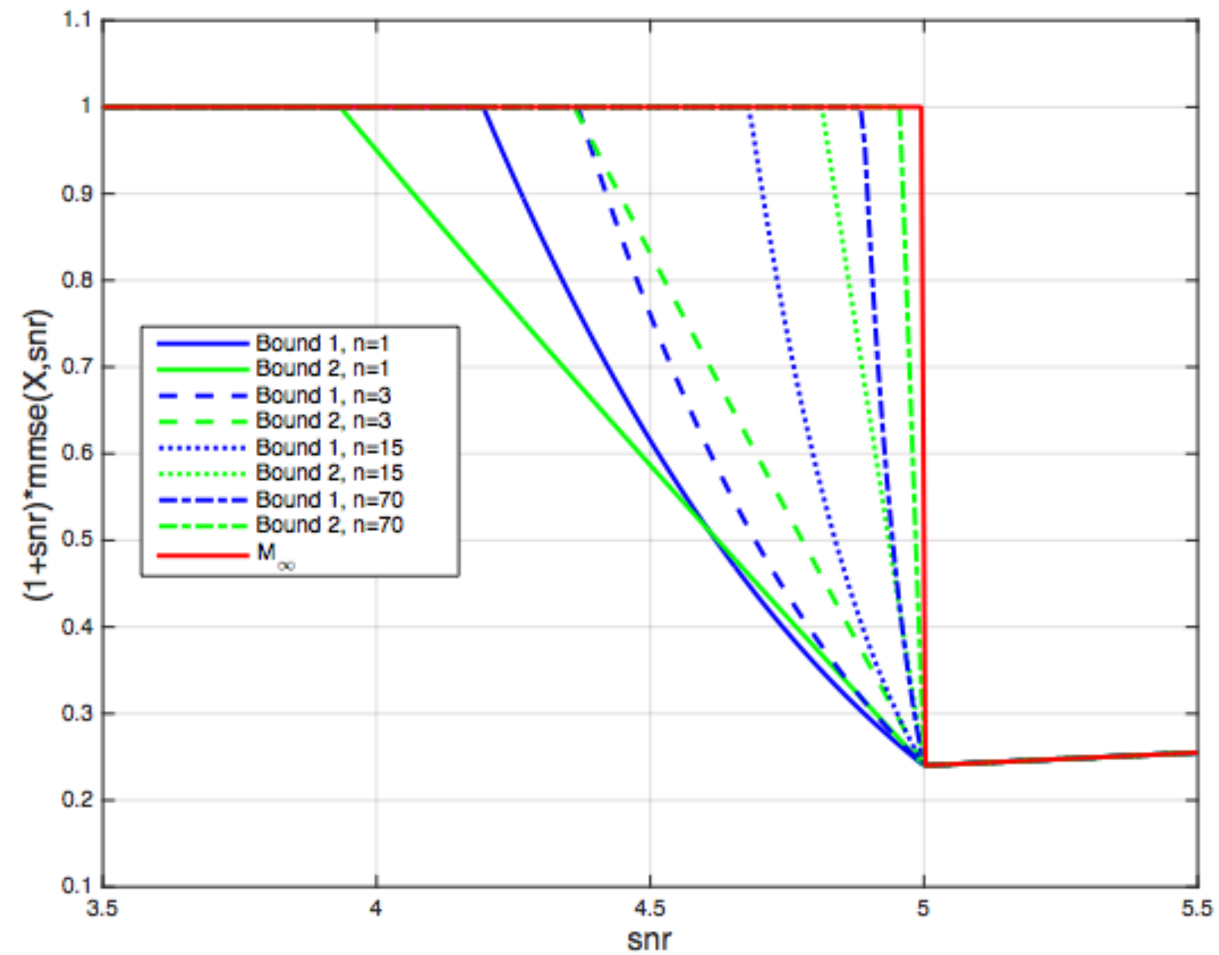
A. Dytso, R. Bustin, D. Tuninetti, N. Devroye, S. Shamai, and H. V. Poor, “On the Minimum Mean p-th Error in Gaussian Noise Channels and its Applications ,” Submitted to *IEEE Trans. Inf. Theory,* https://arxiv.org/pdf/1603.07628, 2016.

New Bound on max-MMSE



snr_0=5, beta=0.01, n=1



snr_0=5, beta=0.05

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max-MMSE problem

max-MMSE problem

$$\mathrm{M}_n(\mathrm{snr}, \mathrm{snr}_0, \beta) := \sup_{\mathbf{X}} \mathrm{mmse}(\mathbf{X}, \mathrm{snr}),$$

$$\text{s.t. } \|\mathbf{X}\|_2^2 \leq 1,$$

$$\text{and } \mathrm{mmse}(\mathbf{X}, \mathrm{snr}_0) \leq \frac{\beta}{1+\beta\mathrm{snr}_0}.$$

The problem is still open for snr<snr0

Conjecture: for n=1 the optimal input distribution is discrete.

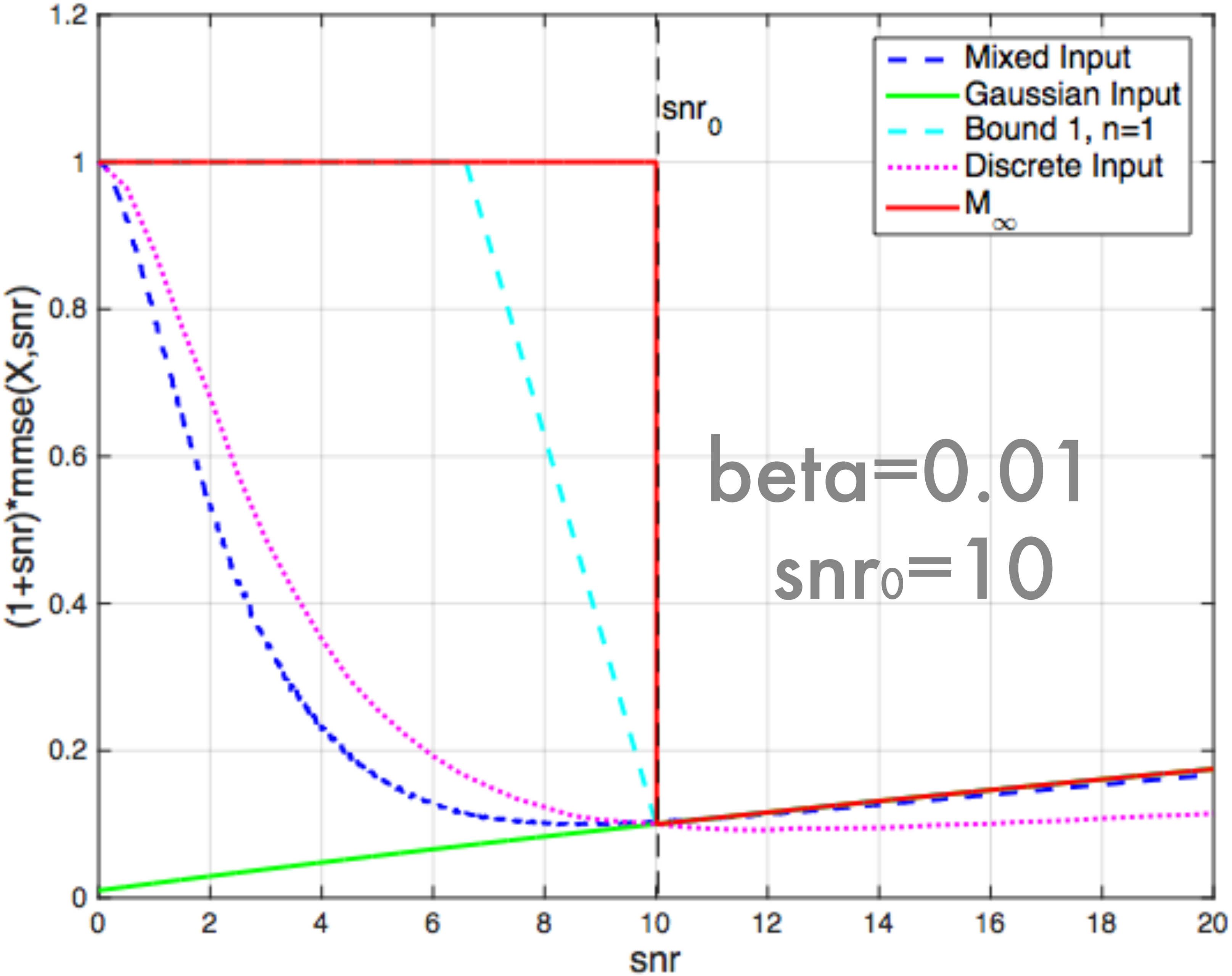
Difficulty: Not a convex optimization problem

Can we give some supporting arguments?

S. Shamai, "From constrained signaling to network interference alignment via an information-estimation perspective," IEEE Information Theory Society Newsletter, vol. 62, no. 7, pp. 6–24, September 2012.

J. G. Smith, "The information capacity of amplitude-and variance- constrained scalar Gaussian channels," *Information and Control*, vol. 18, no. 3, pp. 203–219, 1971.

Bounds for max-MMSE

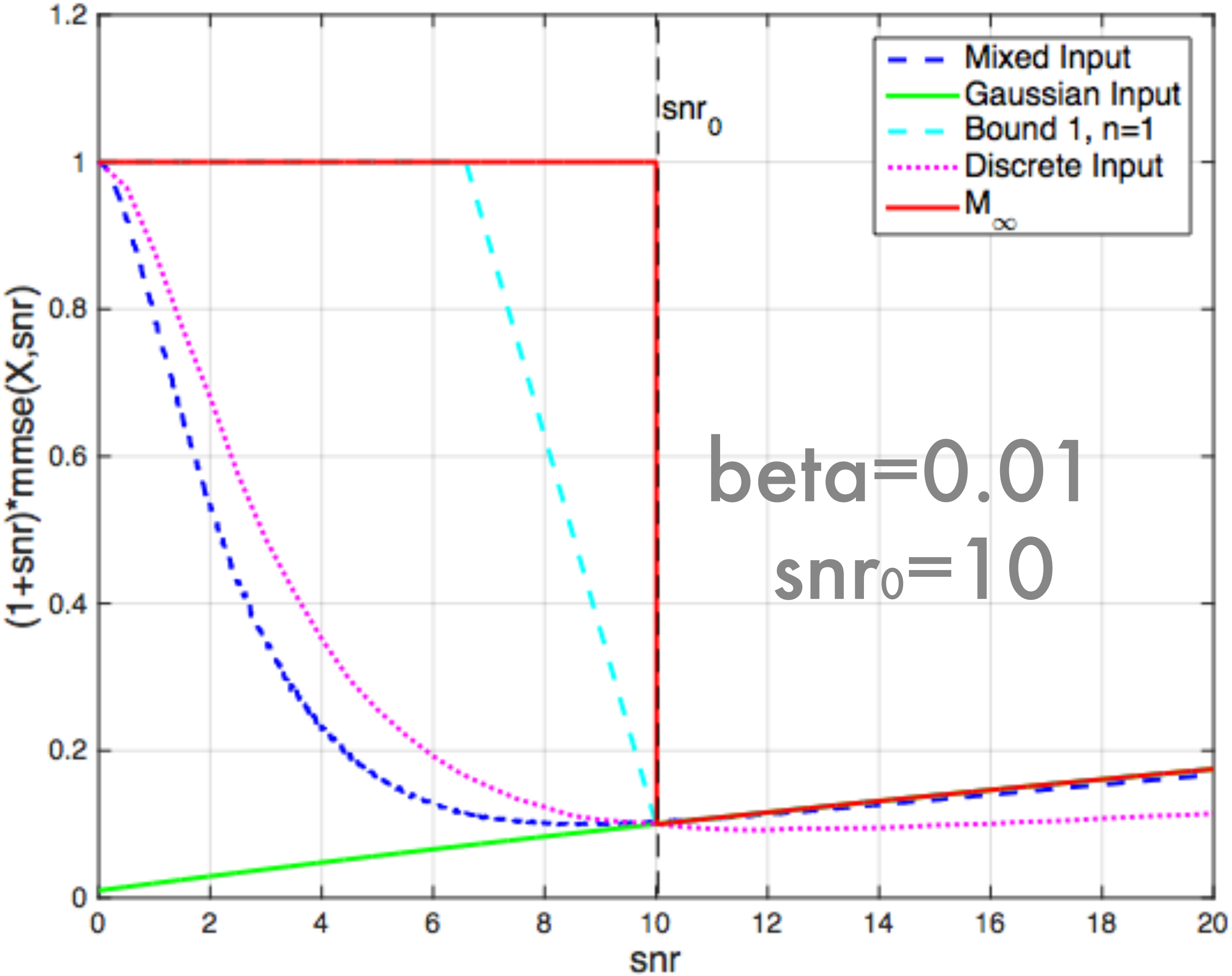


Discrete Input:

$\mathcal{X}_D = [-1.8412, -1.7386, 0.5594]$

with $P_X = [0.1111, 0.1274, 0.7615]$

Bounds for max-MMSE



Discrete Input:

$\mathcal{X}_D = [-1.8412, -1.7386, 0.5594]$
with $P_X = [0.1111, 0.1274, 0.7615]$

Mixed Input:

$$X_{\text{mix}} = \sqrt{1-\delta} X_D + \sqrt{\delta} X_G,$$
$$\delta \in [0,1],$$
$$X_G \sim \mathcal{N}(0,1)$$
$$\mathbb{E}[X_D^2] \leq 1$$

$$\delta = \beta \frac{\text{snr}_0}{1+\text{snr}_0}$$

Mixed Inputs get the best of 'both worlds'

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Summary

- I-MMSE $$I_n(\mathbf{X}, \mathrm{snr}) = \frac{1}{2}\int_0^{\mathrm{snr}} \mathrm{mmse}(\mathbf{X}, t)dt$$
- LMMSE bound $$\mathrm{mmse}(\mathbf{X}, \mathrm{snr}) \leq \frac{1}{1+\mathrm{snr}},$$
- SCPP bound
- Max-MMSE Problem
- Complementary SCPP bounds

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Applications of the I-MMSE

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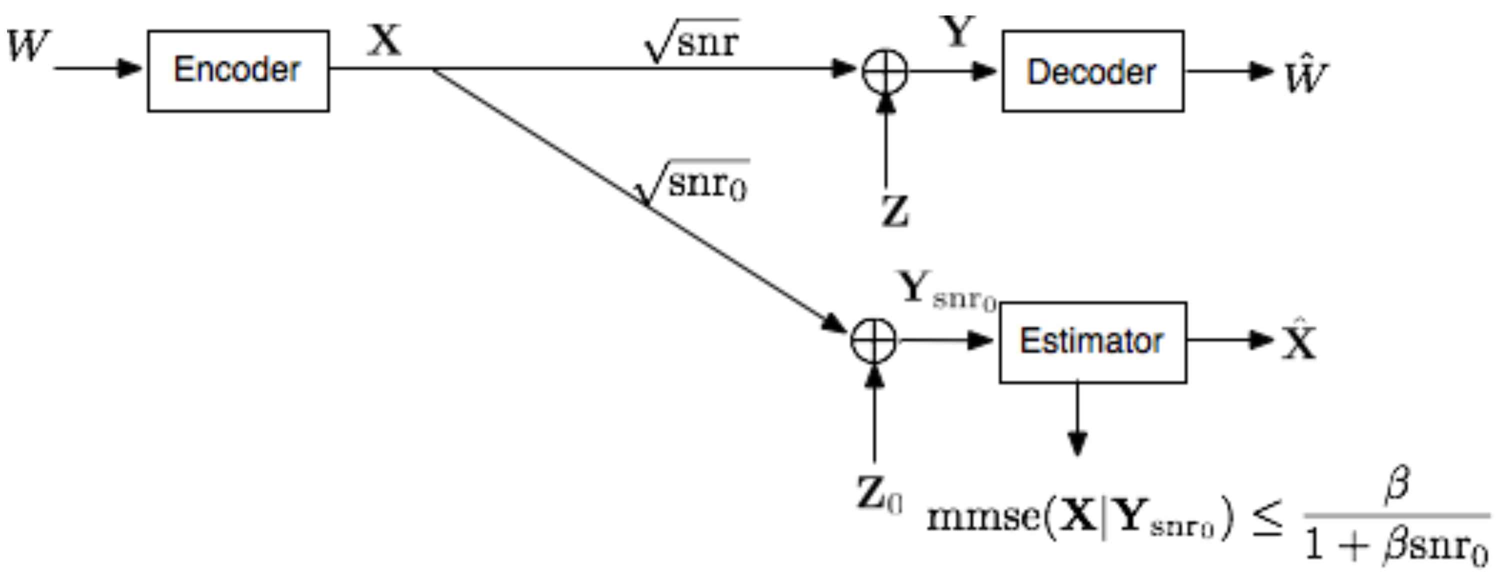
Communication with the Disturbance Constraint



R. Bustin and S. Shamai, "MMSE of 'bad' codes," *IEEE Trans. Inf. Theory*, vol. 59, no. 2, pp. 733–743, Feb 2013.

max-I problem

$$\mathcal{C}_n(\text{snr}, \text{snr}_0, \beta) := \sup_{\mathbf{X}} \frac{1}{n} I(\mathbf{X}, \mathbf{Y}_{\text{snr}}),$$

$$\text{s.t.} \|\mathbf{X}\|_2^2 \le 1,$$

$$\text{and } \text{mmse}(\mathbf{X}, \text{snr}_0) \le \frac{\beta}{1+\beta\text{snr}_0}.$$

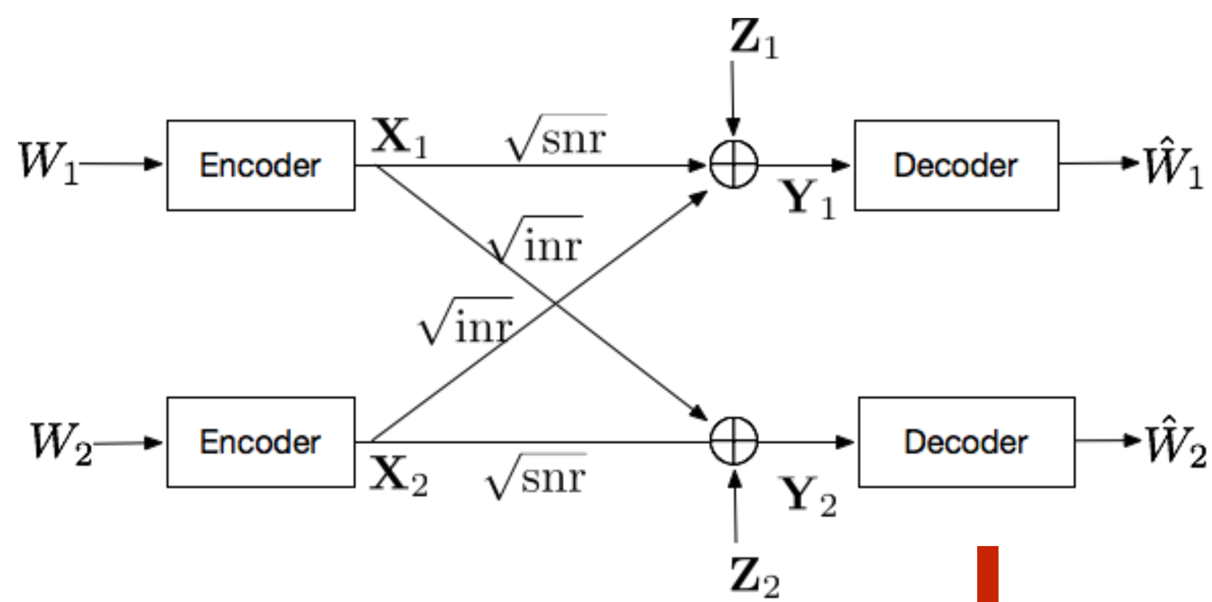
S. Shamai, "From constrained signaling to network interference alignment via an information-estimation perspective," IEEE Information Theory Society Newsletter, vol. 62, no. 7, pp. 6–24, September 2012.

R. Blasco-Serrano, R. Thobaben, and M. Skoglund, "Communication and interference coordination," in *Proc. Workshop on Info. Theory and Applications*. IEEE, 2014, pp. 1–8.

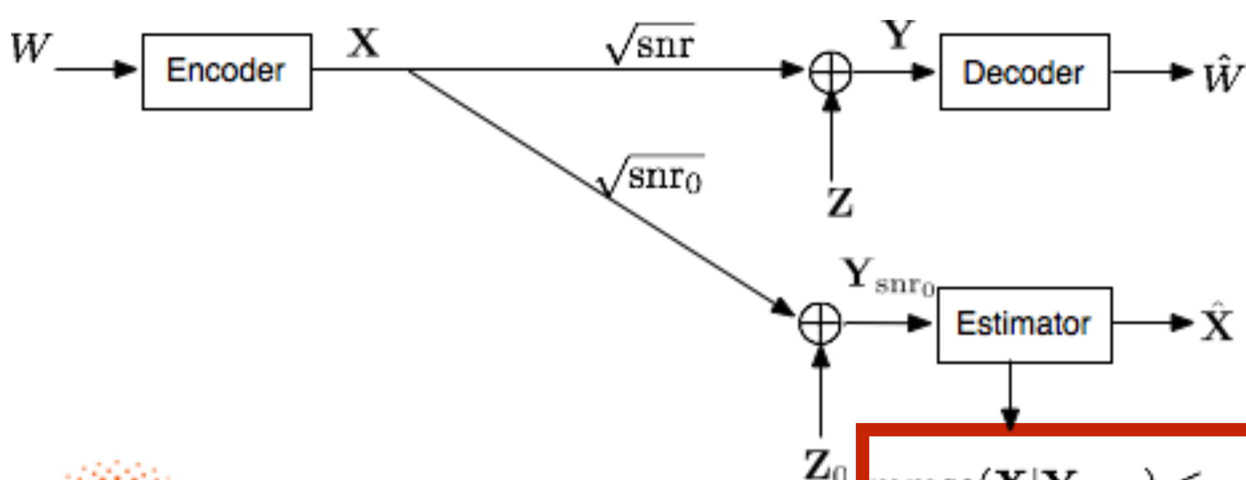
B. Bandemer and A. El Gamal, "Communication with disturbance constraints," *IEEE Trans. Inf. Theory*, vol. 60, no. 8, pp. 4488–4502, Aug 2014.

Motivation

Gaussian Interference Channel



Simplified Model



$$\text{mmse}(\mathbf{X}|\mathbf{Y}_{\text{snr}_0}) \leq \frac{\beta}{1+\beta\text{snr}_0}$$

MMSE Constraint captures the undecodable interference

S. Shamai, "From constrained signaling to network interference alignment via an information-estimation perspective," IEEE Information Theory Society Newsletter, vol. 62, no. 7, pp. 6–24, September 2012.

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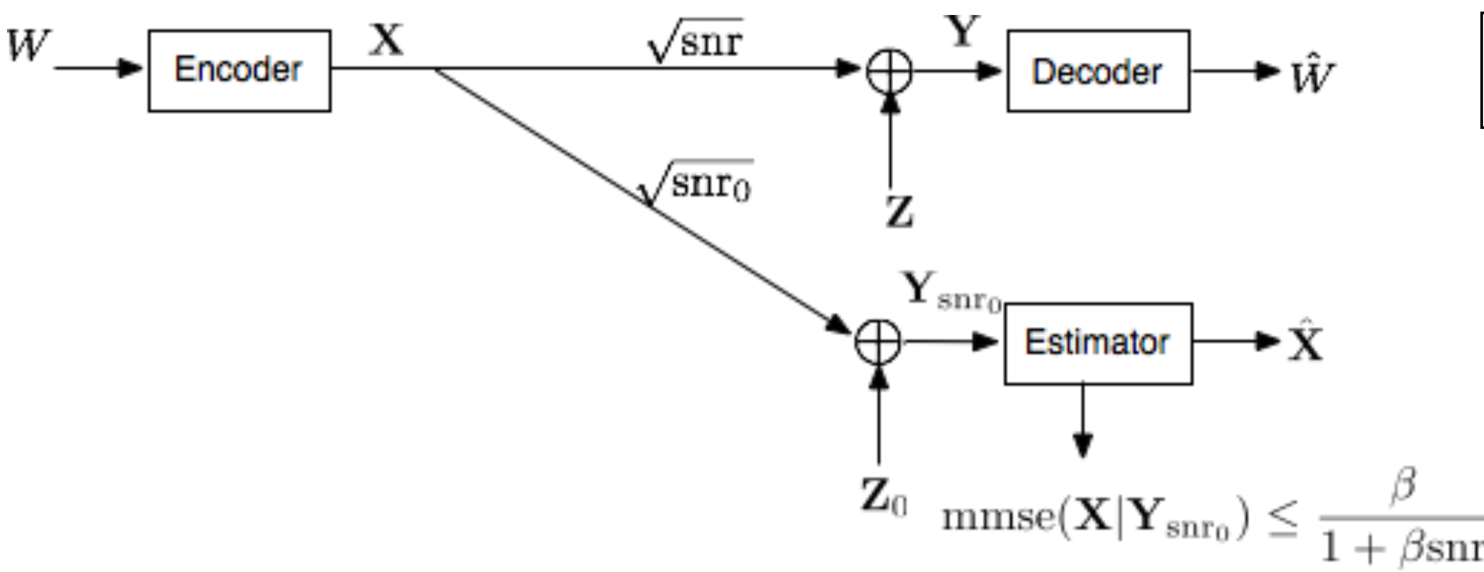
Communication with the Disturbance Constraint



R. Bustin and S. Shamai, "MMSE of 'bad' codes," *IEEE Trans. Inf. Theory*, vol. 59, no. 2, pp. 733–743, Feb 2013.

max-I problem

$$\mathcal{C}_\infty(\mathrm{snr}, \mathrm{snr}_0, \beta) = \lim_{n\to\infty} \mathcal{C}_n(\mathrm{snr}, \mathrm{snr}_0, \beta),$$

$$= \begin{cases} \frac{1}{2}\log(1+\mathrm{snr}), & \mathrm{snr} \le \mathrm{snr}_0, \\ \frac{1}{2}\log(1+\beta\mathrm{snr}) + \frac{1}{2}\log\left(1+\frac{\mathrm{snr}_0(1-\beta)}{1+\beta\mathrm{snr}_0}\right), & \mathrm{snr} \ge \mathrm{snr}_0, \end{cases}$$

Shows that superposition coding is optimal: Supports the Han-Kobayashi approach for the Gaussian interference channel

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Communication with the Disturbance Constraint

Converse

$$
\begin{aligned}
I(\mathbf{X}, \mathrm{snr}) &= \frac{1}{2}\int_0^{\mathrm{snr}} \mathrm{mmse}(\mathbf{X}, t)dt \\
&= \frac{1}{2}\int_0^{\mathrm{snr}_0} \mathrm{mmse}(\mathbf{X}, t)dt + \frac{1}{2}\int_{\mathrm{snr}_0}^{\mathrm{snr}} \mathrm{mmse}(\mathbf{X}, t)dt \\
&\leq \frac{1}{2}\int_0^{\mathrm{snr}_0} \frac{1}{1+t}dt + \frac{1}{2}\int_{\mathrm{snr}_0}^{\mathrm{snr}} \frac{\beta}{1+\beta t}dt \\
&= \frac{1}{2}\log\left(1+\mathrm{snr}_0\right) + \frac{1}{2}\log\left(\frac{1+\beta\mathrm{snr}}{1+\beta\mathrm{snr}_0}\right) \\
&= \frac{1}{2}\log\left(1+\beta\mathrm{snr}\right) + \frac{1}{2}\log\left(1+\frac{(1-\beta)\mathrm{snr}_0}{1+\beta\mathrm{snr}_0}\right)
\end{aligned}
$$

Achievability: Superposition Codes

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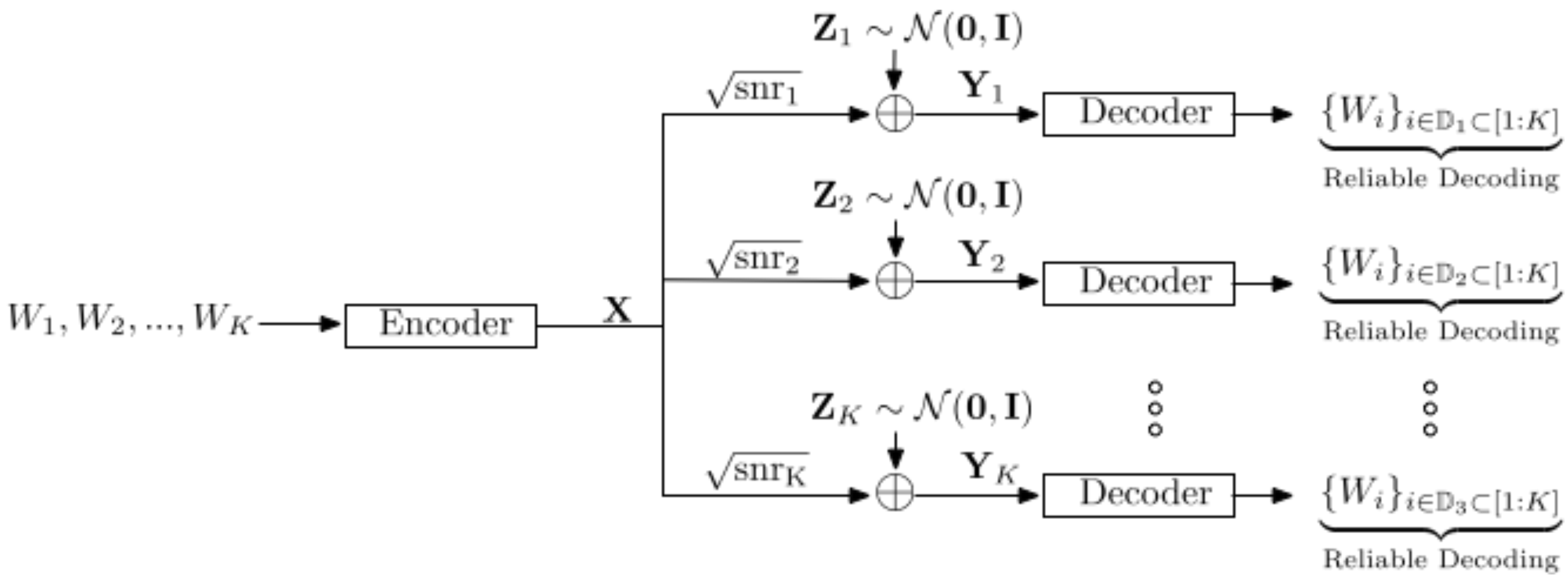
SNR Evolution of the MMSE



Derive new results that give a different graphical interpretation for general code sequences in the scalar Gaussian regime.

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Bustin, R., Schaefer, R. F., Poor, H. V., Shitz, S. S. (2016). On the SNR-evolution of the MMSE function of codes for the Gaussian broadcast and wiretap channels. IEEE Transactions on Information Theory, 62(4), 2070-2091.

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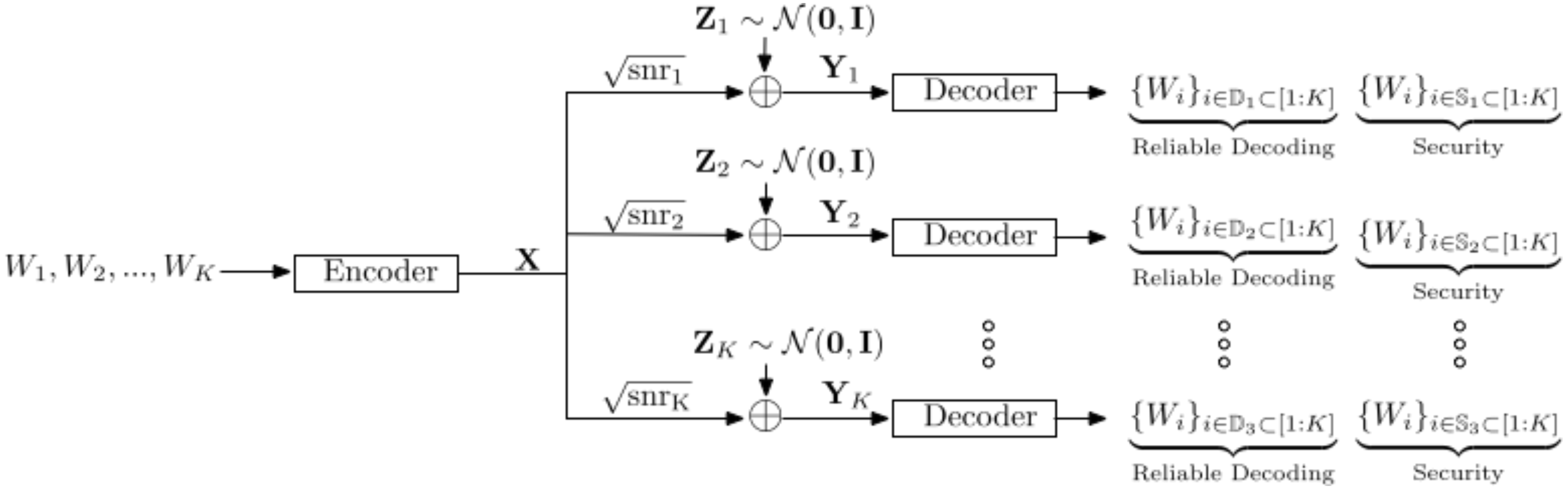

Our technique can also take care of security constraints.

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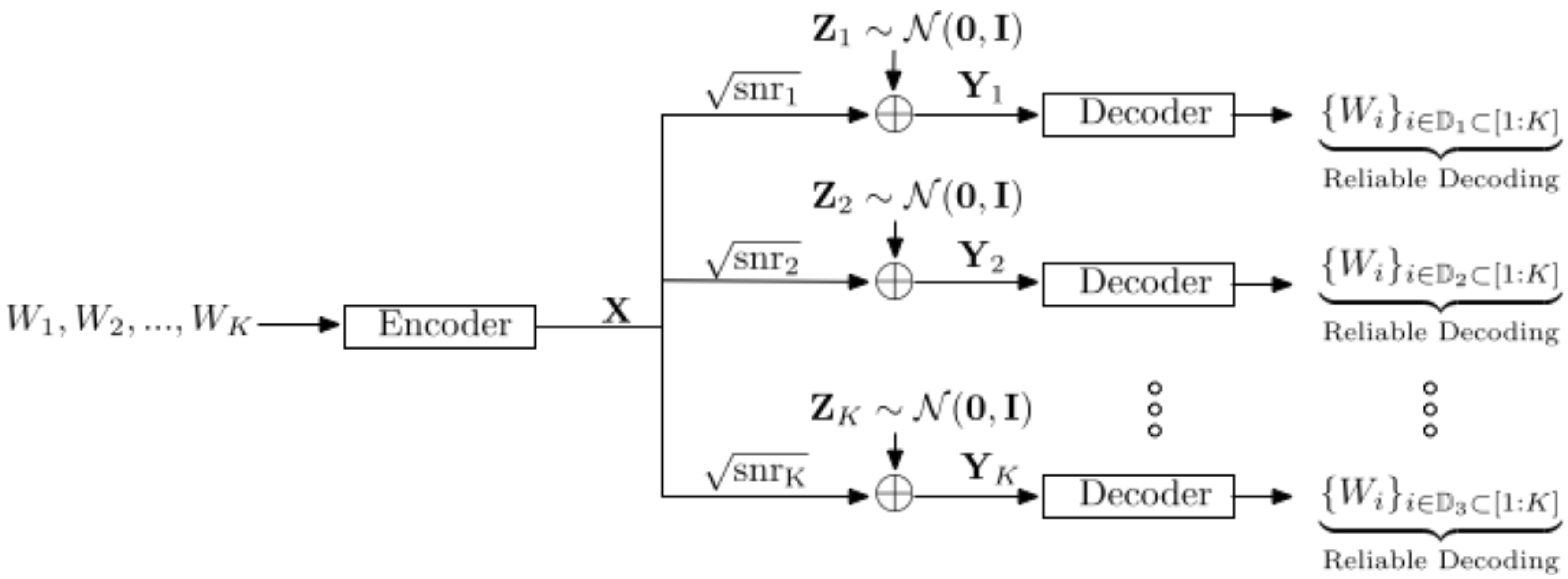
Reliable Comm. Case



Theorem: For a set of independent messages $\{W_1, ..., W_K\}$ such that W_i is reliably decoded at snr_i and $\text{snr}_1 \leq \text{snr}_2 \leq ... \leq \text{snr}_K$ we have that

$$R_i = \frac{1}{2} \int_0^{\text{snr}_i} \text{mmse}\left(\mathbf{X}, \gamma \mid W_1, ..., W_{i-1}\right) - \text{mmse}\left(\mathbf{X}, \gamma \mid W_1, ..., W_i\right) d\gamma.$$

Amazingly!!! We can give an expression for rates in terms of the conditional MMSE.

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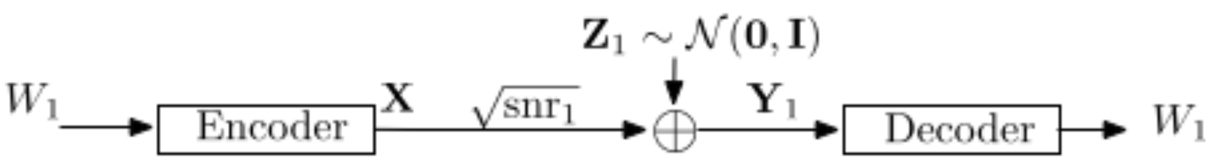
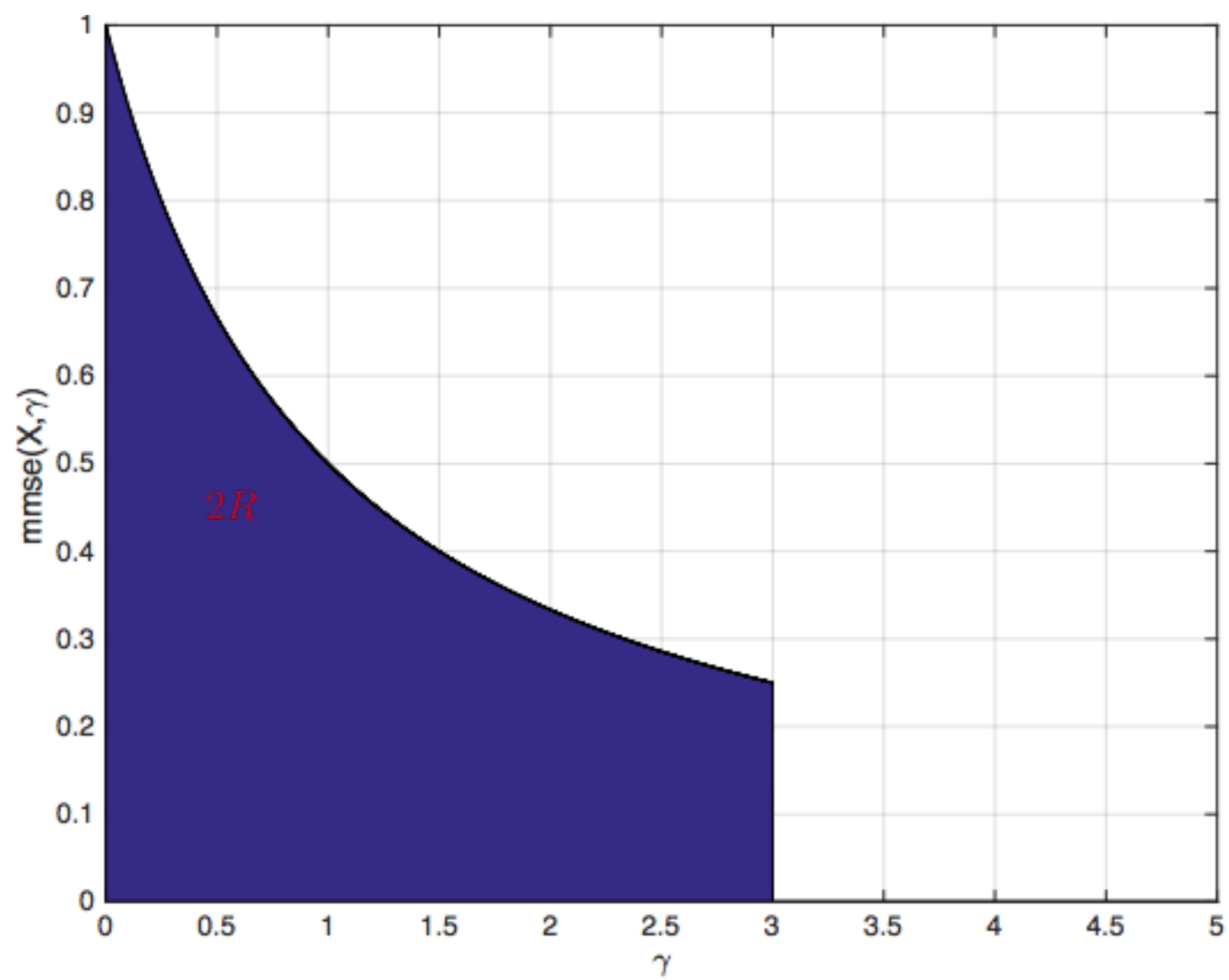
Point-to-Point Channel



$$R = \frac{1}{2}\int_0^{\mathrm{snr}_1} \mathrm{mmse}\left(\mathbf{X}, \gamma\right) - \mathrm{mmse}\left(\mathbf{X}, \gamma \mid W_1\right) d\gamma$$
$$= \frac{1}{2}\int_0^{\mathrm{snr}_1} \mathrm{mmse}\left(\mathbf{X}, \gamma\right) d\gamma.$$

$$\mathrm{mmse}(\mathbf{X}, \gamma) = \begin{cases} \frac{1}{1+\gamma}, & \gamma \leq \mathrm{snr}_1 \\ 0, & \gamma > \mathrm{snr}_1 \end{cases}$$

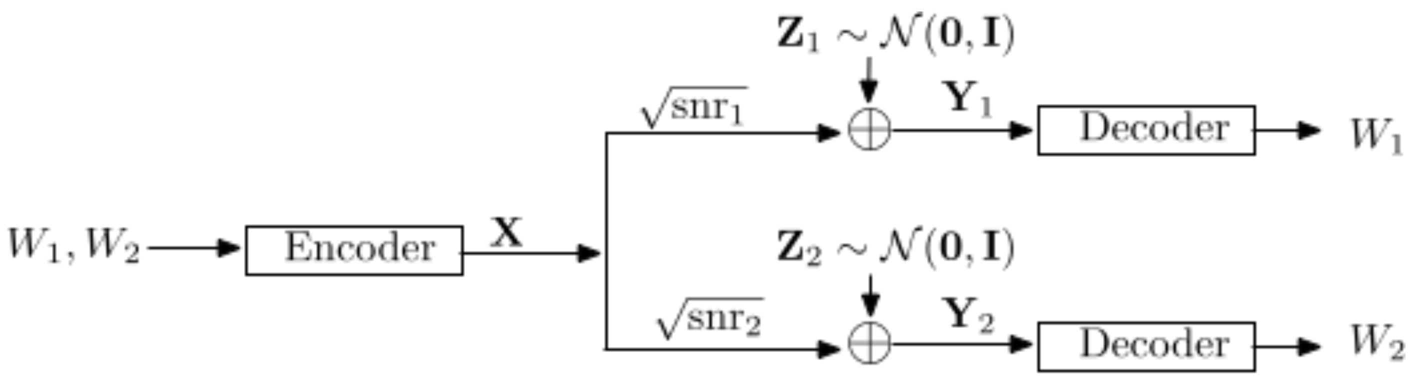
SNR evolution of the MMSE



N. Merhav, D. Guo, and S. Shamai, "Statistical physics of signal estimation in Gaussian noise: Theory and examples of phase transitions," *IEEE Trans. Inf. Theory*, vol. 56, no. 3, pp. 1400–1416, March 2010.

Reliable Comm. Case

Broadcast Channel



$\mathrm{snr}_2 \geq \mathrm{snr}_1$

$$R_1 = \frac{1}{2}\int_0^{\mathrm{snr}_1} \mathrm{mmse}\left(\mathbf{X}, \gamma\right) - \mathrm{mmse}\left(\mathbf{X}, \gamma \mid W_1\right) d\gamma$$

$$R_2 = \frac{1}{2}\int_0^{\mathrm{snr}_2} \mathrm{mmse}\left(\mathbf{X}, \gamma \mid W_1\right) - \mathrm{mmse}\left(\mathbf{X}, \gamma \mid W_1, W_2\right) d\gamma$$

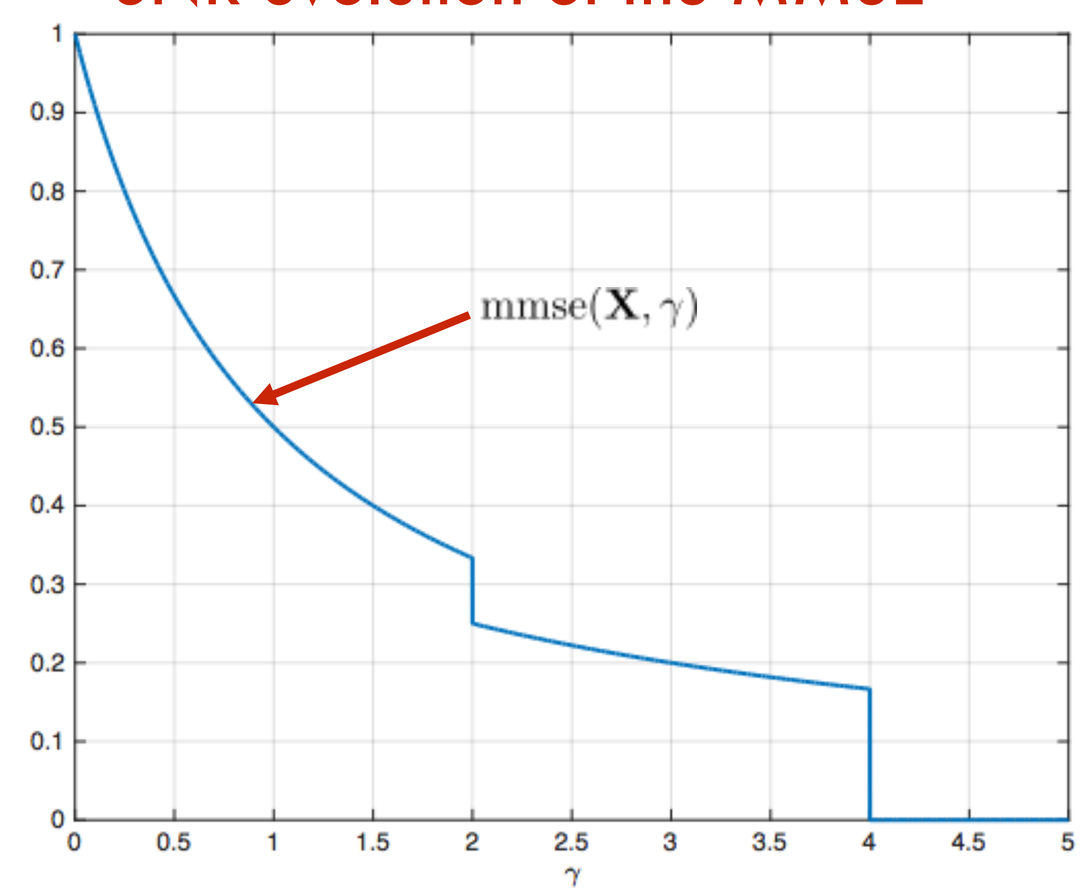
$$= \frac{1}{2}\int_0^{\mathrm{snr}_2} \mathrm{mmse}\left(\mathbf{X}, \gamma \mid W_1\right) d\gamma$$

Moreover,

$$\mathrm{mmse}(\mathbf{X}, \gamma | W_1) = \mathrm{mmse}(\mathbf{X}, \gamma), \forall \gamma > \mathrm{snr}_1.$$

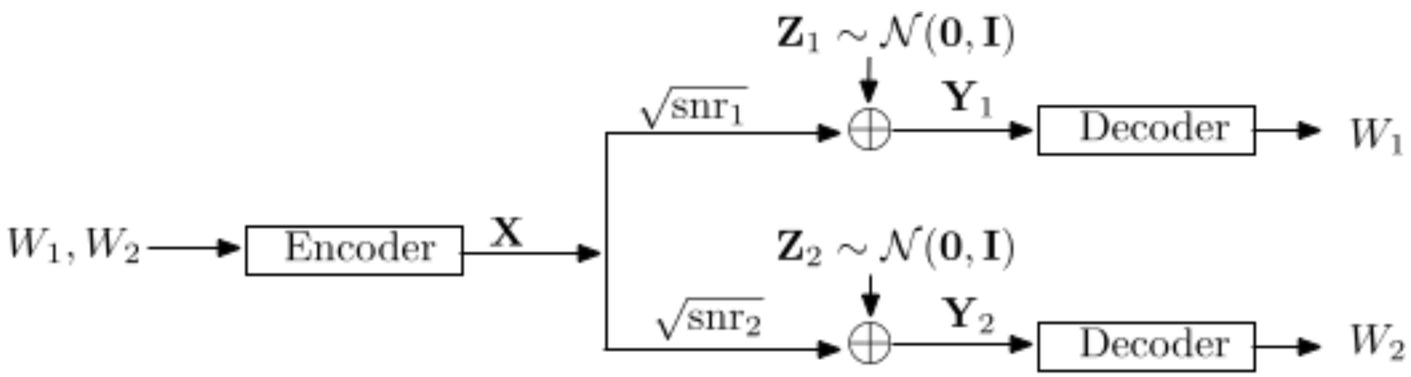
Once W_1 is decoded it provides no information to the estimation of the transmitted codeword.

SNR evolution of the MMSE



Reliable Comm. Case

Broadcast Channel



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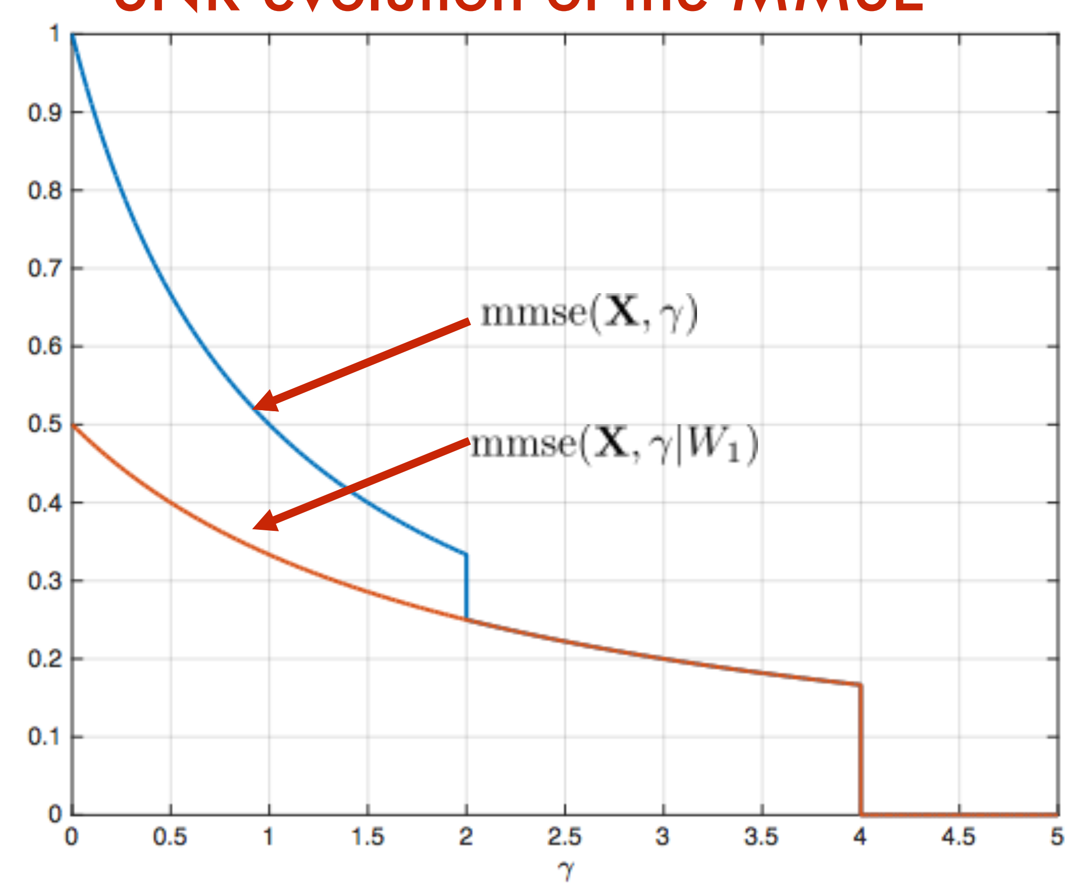

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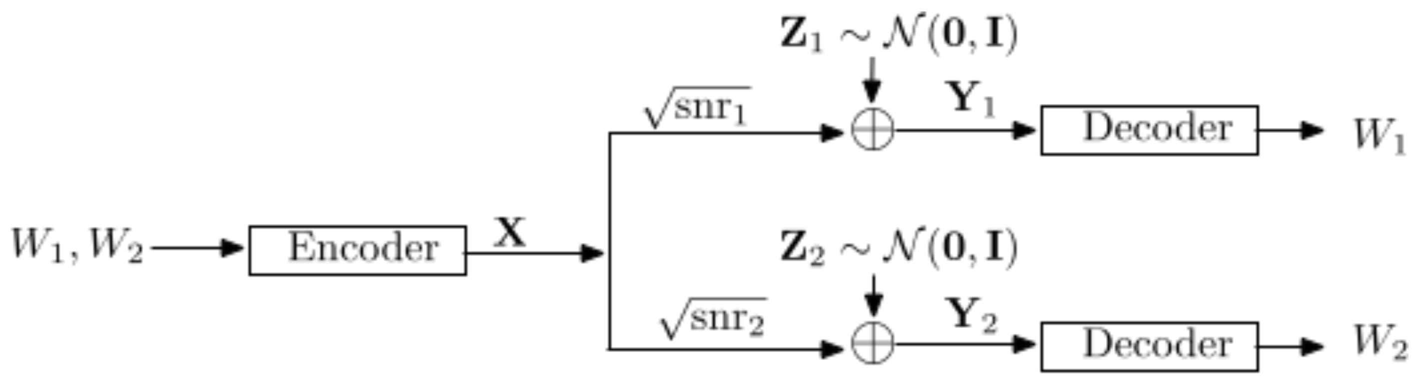
Reliable Comm. Case

Broadcast Channel



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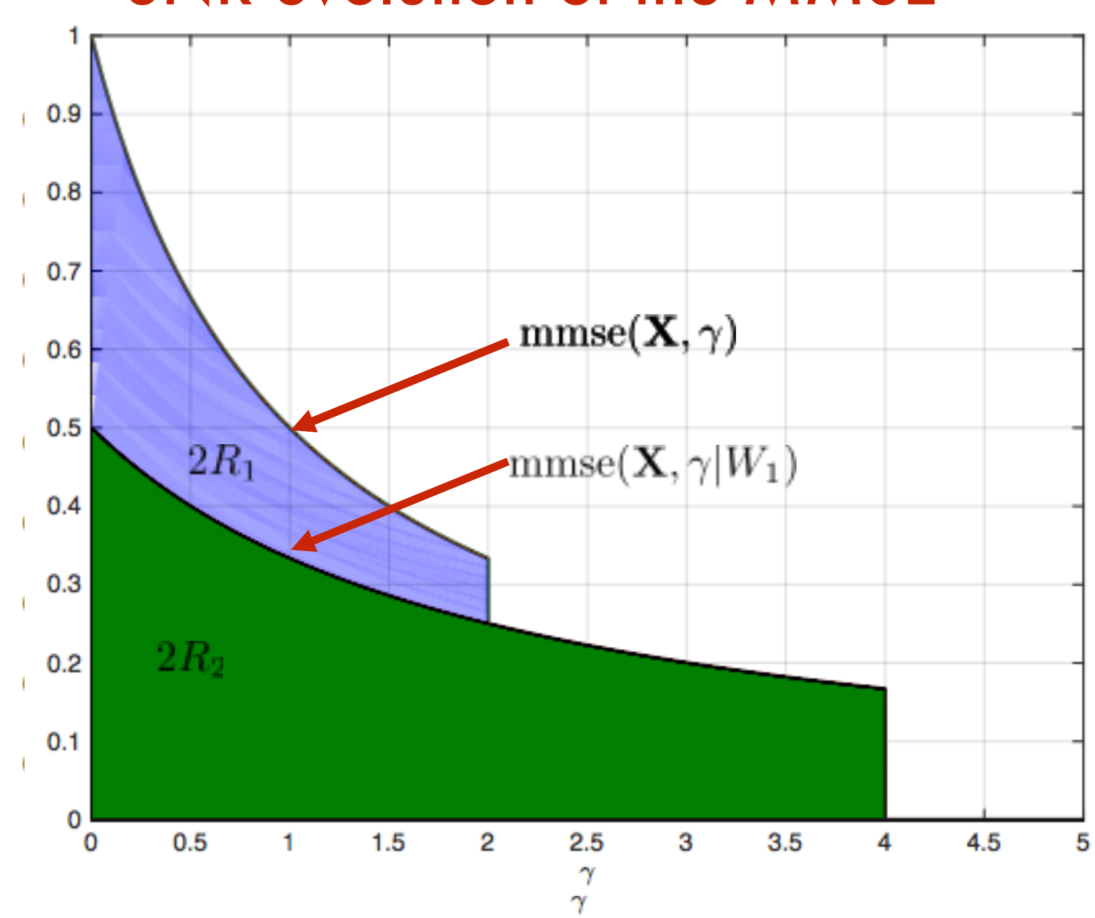

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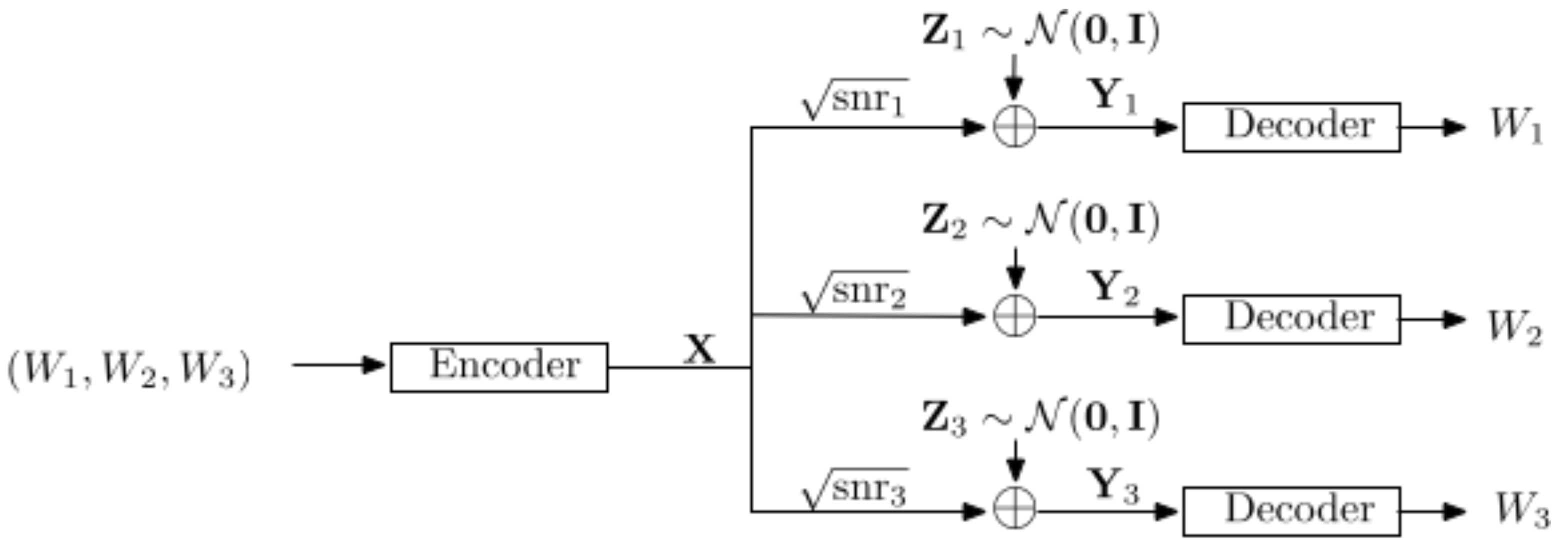
Broadcast Channel



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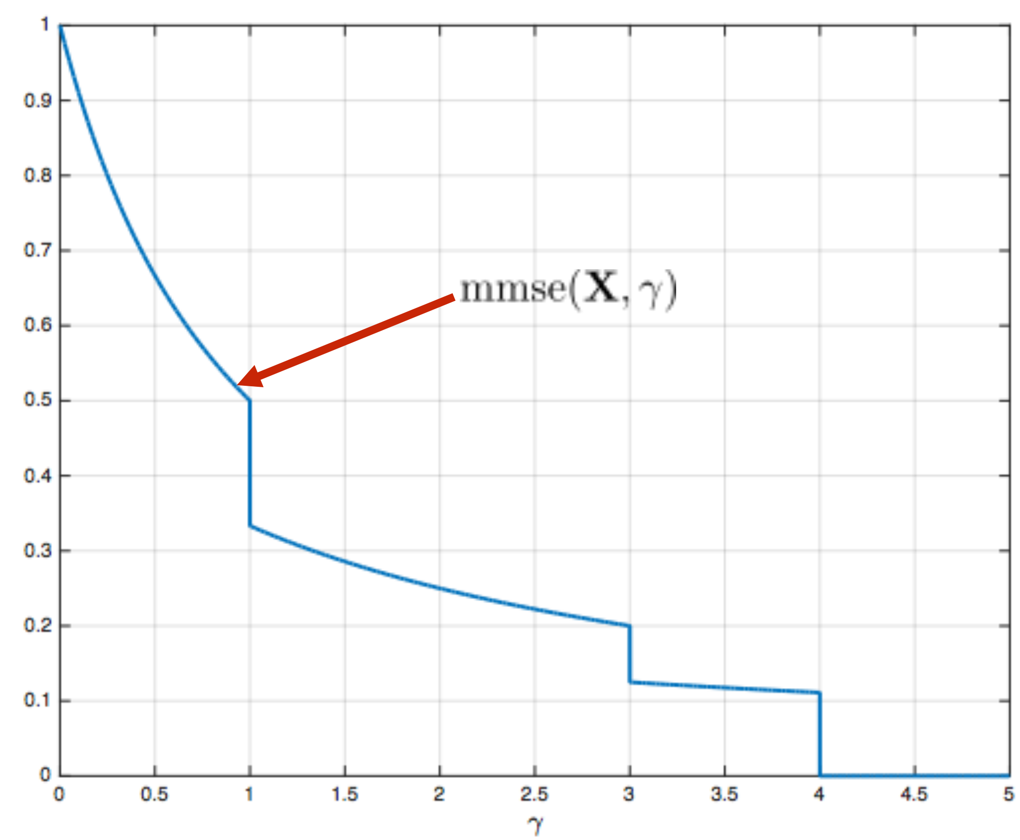
SNR evolution of the MMSE



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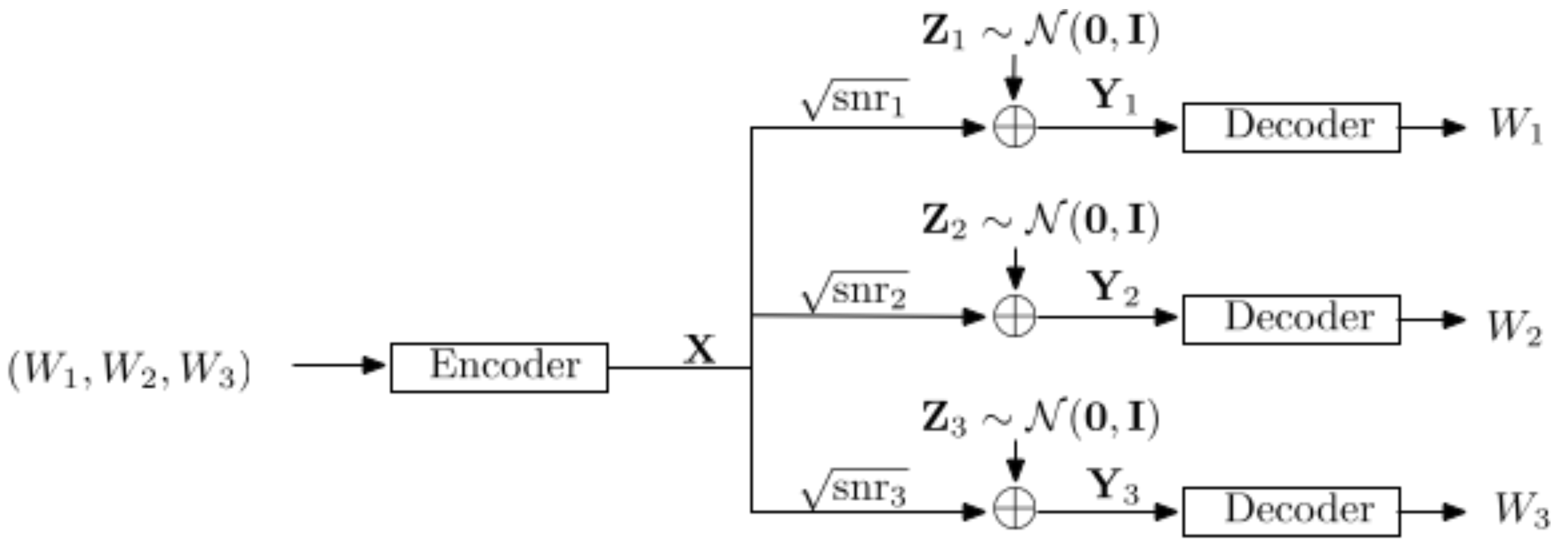
Broadcast Channel



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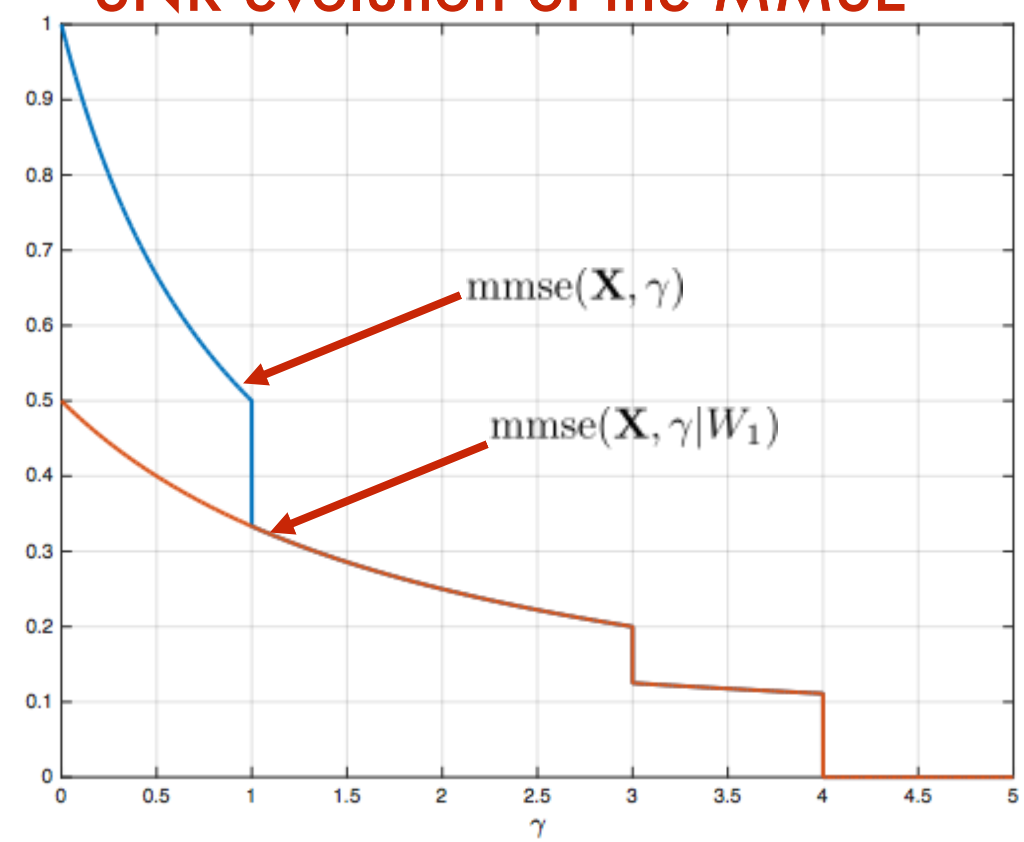

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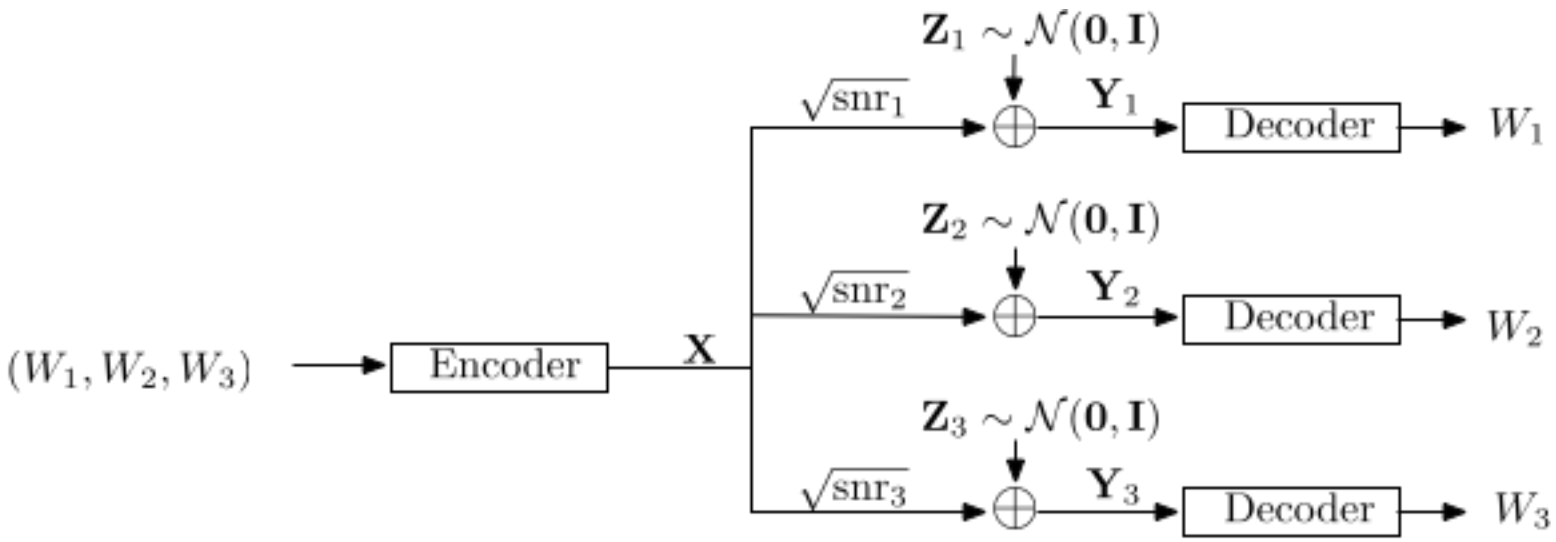

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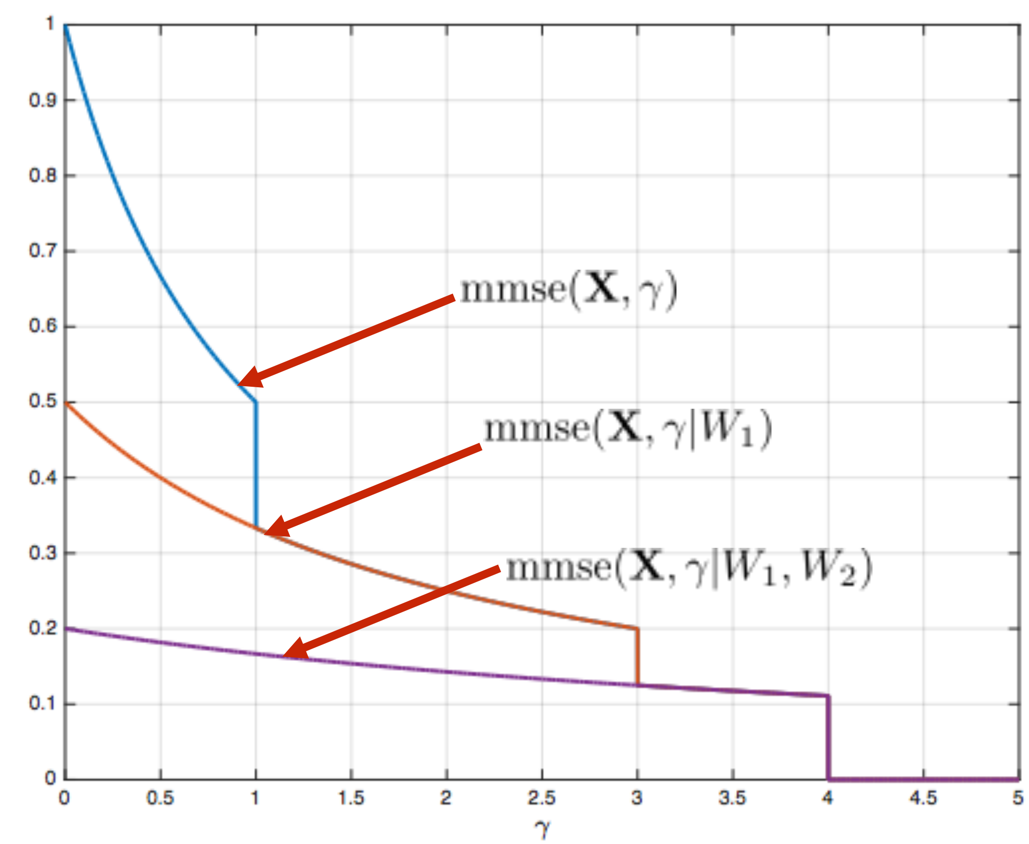

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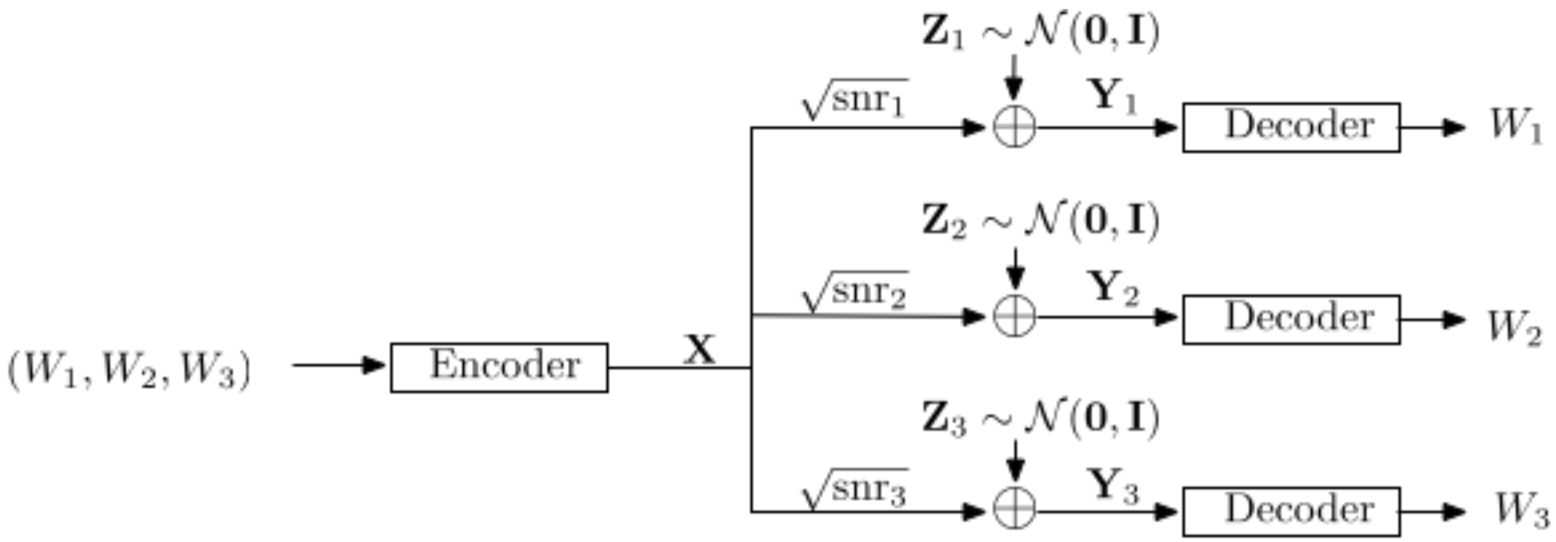

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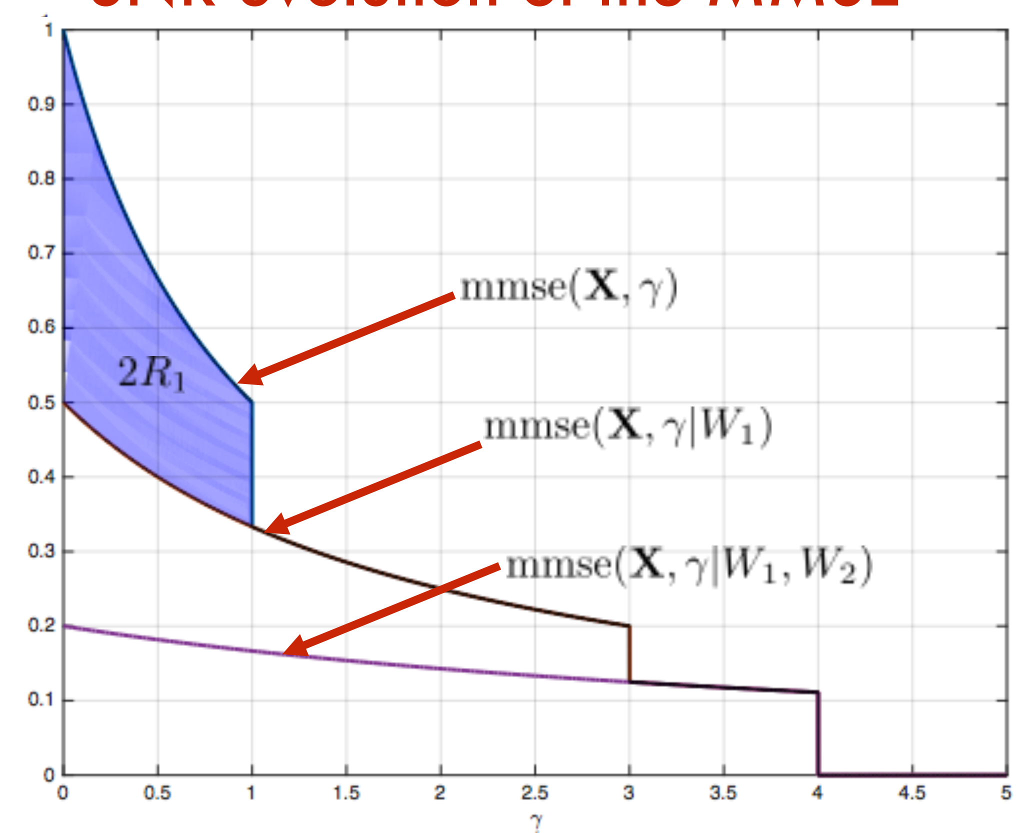

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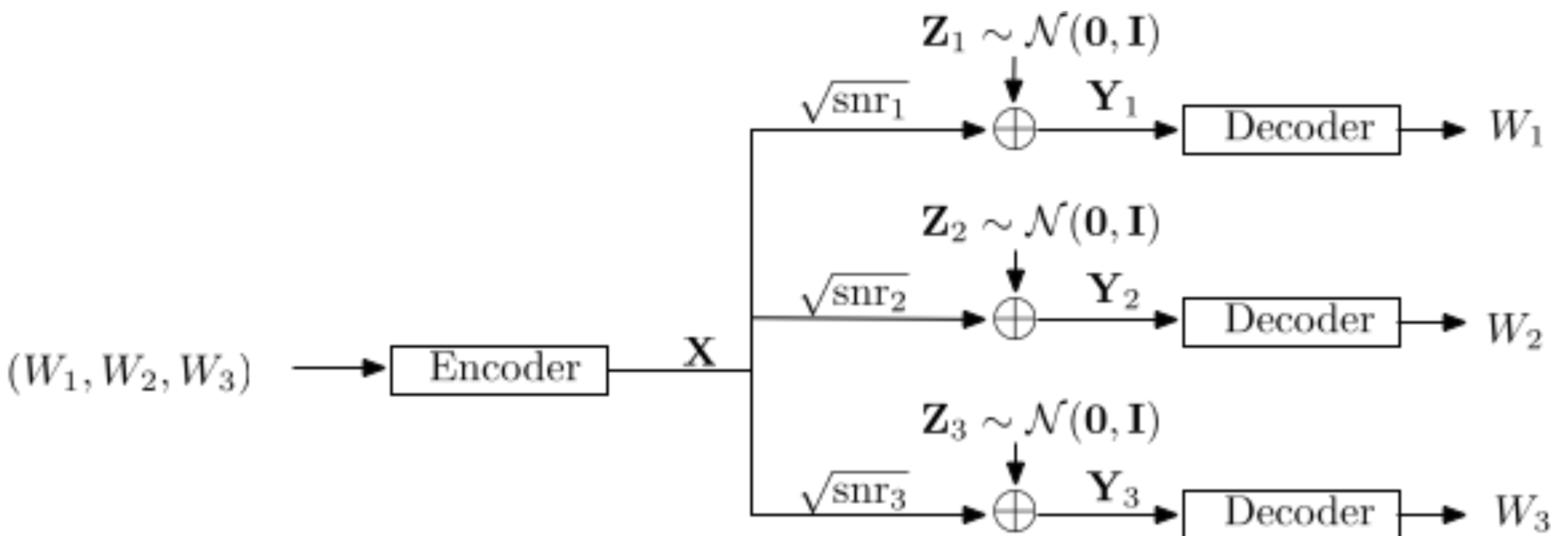

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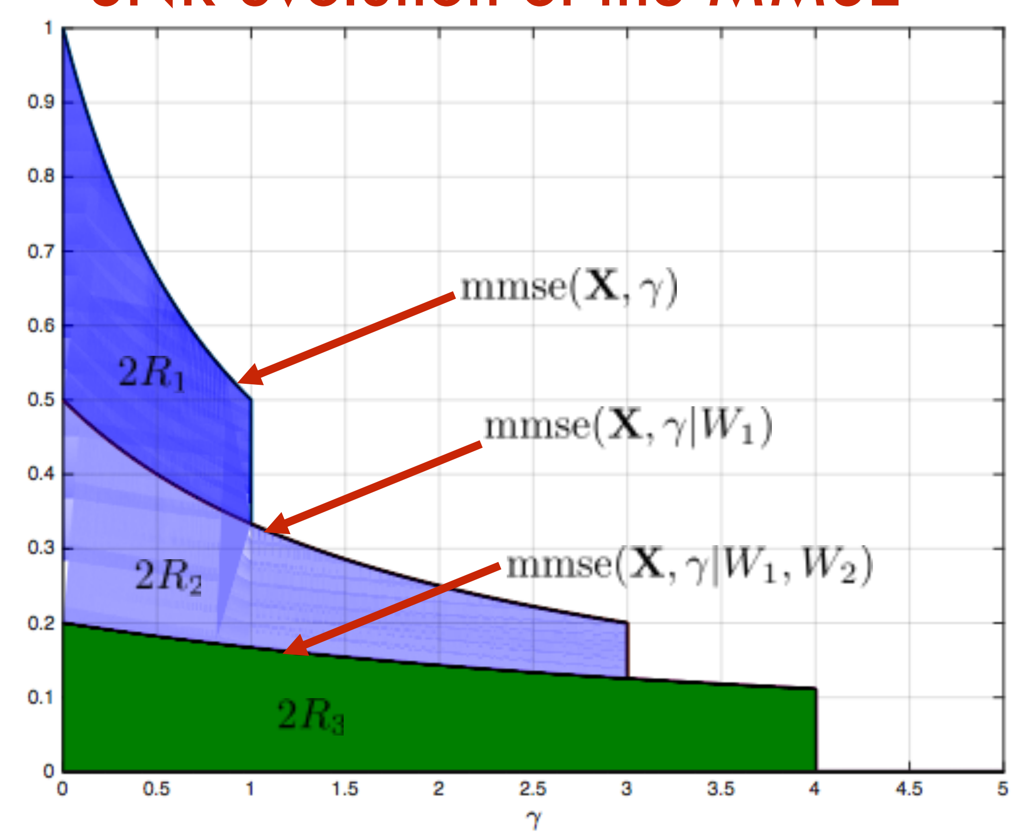

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Applications of the MMPE

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Entropy-Moment Inequality

Theorem: For any $\mathbf{U} \in \mathbb{R}^n$ and $\|\mathbf{U}\|_p < \infty$ for some $p \in (0, \infty)$ we have

$$h(\mathbf{U}) \leq \frac{n}{2} \log \left(k_{n,p}^2 \cdot n^{\frac{2}{p}} \cdot \|\mathbf{U}\|_p^2 \right),$$

where

$$k_{n,p} := \frac{\sqrt{\pi} \left(\frac{p}{n}\right)^{\frac{1}{p}} \mathrm{e}^{\frac{1}{p}} \Gamma^{\frac{1}{n}} \left(\frac{n}{p} + 1\right)}{\Gamma^{\frac{1}{n}} \left(\frac{n}{2} + 1\right)} = \sqrt{2\pi \mathrm{e}} \frac{1}{n^{\frac{1}{2}} \left(\frac{p}{2}\right)^{\frac{1}{2n}}} + o\left(\frac{n}{p}\right),$$

achieved iff $\mathbf{U}$ is generalized Gaussian (i.e., $\mathrm{e}^{-|\mathbf{u}|^p}$).

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Entropy-Moment Inequality

Conditional Version

Theorem: For any $\mathbf{U} \in \mathbb{R}^n$ such that $h(\mathbf{U}) < \infty$ and $\|\mathbf{U}\|_p < \infty$ for some $p \in (0, \infty)$, and for any $\mathbf{V} \in \mathbb{R}^n$, we have

$$h(\mathbf{U}|\mathbf{V}) \leq \frac{n}{2} \log \left(k_{n,p}^2 \cdot n^{\frac{2}{p}} \cdot \mathrm{mmpe}^{\frac{2}{p}}(\mathbf{U}|\mathbf{V}; p) \right).$$

Generalization of a continuous analog of Fano's inequality

$$h(\mathbf{U}|\mathbf{V}) \leq \frac{n}{2} \log \left(2\pi \mathrm{e} \cdot n \cdot \mathrm{mmse}(\mathbf{U}|\mathbf{V}) \right)$$

T. Cover and J. Thomas, Elements of Information Theory: Second Edition. Wiley, 2006.

Ozarow-Wyner Bound

(Ozarow-Wyner Bound.) Let X_D be a discrete random variable then

$$H(X_D) - \text{gap} \le I(X_D;Y) \le H(X_D),$$

$$\text{gap} = \frac{1}{2}\log\left(\frac{\pi e}{6}\right) + \frac{1}{2}\log\left(1 + \frac{12 \cdot \text{lmmse}(X_D|Y)}{d_{\min}^2(X_D)}\right),$$

$$\text{lmmse}(X_D|Y) = \frac{\mathbb{E}[X_D^2]}{1 + \text{snr} \cdot \mathbb{E}[X_D^2]}$$

$$d_{\min}(X_D) = \inf_{x_i, x_j \in \text{supp}(X_D): i \neq j} |x_i - x_j|.$$

L. Ozarow and A. Wyner, “On the capacity of the Gaussian channel with a finite number of input levels,” *IEEE Trans. Inf. Theory*, vol. 36, no. 6, pp. 1426–1428, Nov 1990.

$$\text{mmse}(X_D|Y) \le \text{lmmse}(X_D|Y),$$

$$\text{lmmse}(X_D|Y) = O\left(\frac{1}{\text{snr}}\right),$$

$$\text{mmse}(X_D|Y) = O\left(\mathrm{e}^{-\frac{\text{snr} d_{\min}^2}{4}}\right).$$

$$\text{gap} = \frac{1}{2}\log\left(\frac{\pi e}{6}\right) + \frac{1}{2}\log\left(1 + \frac{12 \cdot \text{mmse}(X_D|Y)}{d_{\min}^2(X_D)}\right),$$

$$\text{mmse}(X_D|Y) = \mathbb{E}\left[(X_D - \mathbb{E}[X_D|Y])^2\right].$$

A. Dytso, D. Tuninetti, and N. Devroye, “Interference as noise: Friend or foe?” *IEEE Trans. Inf. Theory*, vol. 62, no. 6, pp. 3561–3596, 2016.

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Simple Application of OW

Gaussian Noise Channel

Take X_D to be PAM with $N \approx \sqrt{1+\text{snr}}$. Then,

$$H(X_D) \approx \frac{1}{2}\log(1+\text{snr}),$$

and we are interested in

$$\frac{1}{2}\log(1+\text{snr}) - I(X_D;Y) = \text{gap}$$

Ozarow and Wyner showed that: $\text{gap} \approx 0.75$

Observed before:

G. Ungerboeck, “Channel coding with multilevel/phase signals,” *IEEE Trans. Inf. Theory*, vol. 28, no. 1, pp. 55–67, 1982.

G.C.E. Shannon. “Systems Which Approach the Ideal as P/N → ∞” March 29, 1948. Unpublished, mentioned (reference [35]), in Shannon’s Collected Papers: Edited: Sloane-Wyner, 1993.

Amazingly! Not Equiprobable!!

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Amazingly! Not Equiprobable!!

Can this be improved?

Next, we provide a bound a generalization of OW bound

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Generalized OW Bound

Theorem: Let $\mathbf{X}_D$ be a discrete random vector with finite entropy, and let $\mathcal{K}_p$ be a set of continuous random vectors, independent of $\mathbf{X}_D$, such that for every $\mathbf{U} \in \mathcal{K}_p$, $h(\mathbf{U}), \|\mathbf{U}\|_p < \infty$, and

$$H(\mathbf{X}_D | \mathbf{X}_D + \mathbf{U}) = 0, \ \forall \mathbf{U} \in \mathcal{K}_p.$$

Then for any $p > 0$

$$[H(\mathbf{X}_D) - \mathsf{gap}_p]^+ \leq I(\mathbf{X}_D; \mathbf{Y}) \leq H(\mathbf{X}_D),$$

where

$$n^{-1}\mathsf{gap}_p \leq \inf_{\mathbf{U} \in \mathcal{K}_p} \left(G_{1,p}(\mathbf{U}, \mathbf{X}_D) + G_{2,p}(\mathbf{U}) \right),$$

$$G_{1,p}(\mathbf{U}, \mathbf{X}_D) = \log \left(\frac{\|\mathbf{U} + \mathbf{X}_D - f_p(\mathbf{X}_D|\mathbf{Y})\|_p}{\|\mathbf{U}\|_p} \right) \leq \begin{cases} \log \left(1 + \frac{\mathrm{mmpe}^{\frac{1}{p}}(\mathbf{X}_D, \mathrm{snr}, p)}{\|\mathbf{U}\|_p} \right), p \neq 2 \\ \frac{1}{2} \log \left(1 + \frac{\mathrm{mmse}(\mathbf{X}_D, \mathrm{snr})}{\|\mathbf{U}\|_2^2} \right), p = 2 \end{cases},$$

$$G_{2,p}(\mathbf{U}) = \log \left(\frac{k_{n,p} \cdot n^{\frac{1}{p}} \cdot \|\mathbf{U}\|_p}{e^{\frac{1}{n} h_e(\mathbf{U})}} \right).$$

A. Dytso, R. Bustin, D. Tuninetti, N. Devroye, S. Shamai, and H. V. Poor, "On the Minimum Mean p-th Error in Gaussian Noise Channels and its Applications," Submitted to *IEEE Trans. Inf. Theory,* https://arxiv.org/pdf/1603.07628, 2016.

A. Dytso, M. Goldenbaum*, H. V. Poor*, and S. Shamai, "A Generalized Ozarow-Wyner Capacity Bound with Applications," Submitted to *Proc. IEEE Int. Symp. Inf. Theory*, 2017

Generalized OW Bound

Proof:

Additive Dither

$$I(\mathbf{X}_D;\mathbf{Y}) \geq I(\mathbf{X}_D+\mathbf{U};\mathbf{Y}), \text{ data processing } \quad \mathbf{X}_D+\mathbf{U} \leftrightarrow \mathbf{X}_D \leftrightarrow \mathbf{Y}$$

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Generalized OW Bound

Proof:

$$I(\mathbf{X}_D;\mathbf{Y}) \geq I(\mathbf{X}_D+\mathbf{U};\mathbf{Y}), \text{ data processing}$$
$$= h(\mathbf{X}_D+\mathbf{U}) - h(\mathbf{X}_D+\mathbf{U}|\mathbf{Y})$$

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Generalized OW Bound

Proof:

$$\begin{aligned} I(\mathbf{X}_D;\mathbf{Y}) &\geq I(\mathbf{X}_D+\mathbf{U};\mathbf{Y}), \text{ data processing} \\ &= h(\mathbf{X}_D+\mathbf{U}) - h(\mathbf{X}_D+\mathbf{U}|\mathbf{Y}) \\ &= H(\mathbf{X}_D) + h(\mathbf{U}) - h(\mathbf{X}_D+\mathbf{U}|\mathbf{Y}), \text{ compact support of } \mathbf{U}, \end{aligned}$$

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Generalized OW Bound

Proof:

$$\begin{aligned} I(\mathbf{X}_D;\mathbf{Y}) &\geq I(\mathbf{X}_D+\mathbf{U};\mathbf{Y}), \text{ data processing} \\ &= h(\mathbf{X}_D+\mathbf{U}) - h(\mathbf{X}_D+\mathbf{U}|\mathbf{Y}) \\ &= H(\mathbf{X}_D) + h(\mathbf{U}) - h(\mathbf{X}_D+\mathbf{U}|\mathbf{Y}), \text{ compact support of } \mathbf{U}, \end{aligned}$$

Next, observe

$$n^{-1}h(\mathbf{X}_D+\mathbf{U}|\mathbf{Y}) \leq \log\left(k_{n,p}\cdot n^{\frac{1}{p}}\cdot\|\mathbf{X}_D+\mathbf{U}-g(\mathbf{Y})\|_p\right)$$

Generalized OW Bound

Proof:

$$
\begin{aligned}
I(\mathbf{X}_D;\mathbf{Y}) &\geq I(\mathbf{X}_D+\mathbf{U};\mathbf{Y}), \text{ data processing} \\
&= h(\mathbf{X}_D+\mathbf{U}) - h(\mathbf{X}_D+\mathbf{U}|\mathbf{Y}) \\
&= H(\mathbf{X}_D) + h(\mathbf{U}) - h(\mathbf{X}_D+\mathbf{U}|\mathbf{Y}), \text{ compact support of } \mathbf{U},
\end{aligned}
$$

Next, observe

$$
n^{-1} h(\mathbf{X}_D+\mathbf{U}|\mathbf{Y}) \leq \log\left(k_{n,p}\cdot n^{\frac{1}{p}}\cdot \|\mathbf{X}_D+\mathbf{U}-g(\mathbf{Y})\|_p\right)
$$

By combining all the bounds

$$
I(\mathbf{X}_D;\mathbf{Y}) \geq H(\mathbf{X}_D) - n\cdot\log\left(\frac{\|\mathbf{U}+\mathbf{X}_D-g(\mathbf{Y})\|_p}{\|\mathbf{U}\|_p}\right) - n\cdot\log\left(\frac{k_{n,p}\cdot n^{\frac{1}{p}}\cdot\|\mathbf{U}\|_p}{\mathrm{e}^{\frac{1}{n}h_e(\mathbf{U})}}\right).
$$

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Simple Application of OW

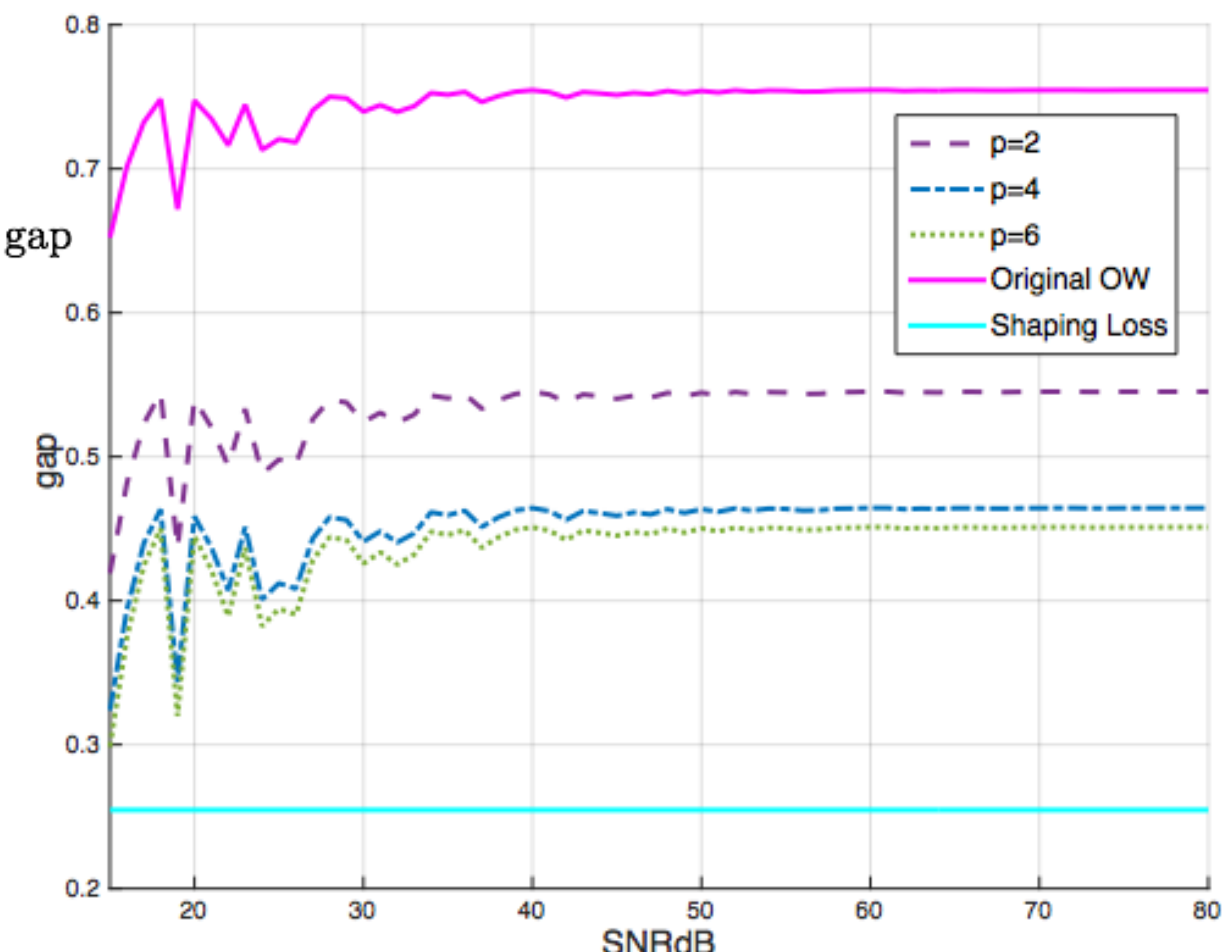
Gaussian Noise Channel

Take X_D to be PAM with $N \approx \sqrt{1+\mathrm{snr}}$. Then,

$$H(X_D) \approx \frac{1}{2}\log(1+\mathrm{snr}),$$

and we are interested in

$$\frac{1}{2}\log(1+\mathrm{snr}) - I(X_D;Y) = \mathrm{gap}$$



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Simple Application of OW

Cauchy Noise Channel

$$Y = X + Z,$$

$$p_Z(z) = \frac{1}{\pi\gamma}\left(\frac{\gamma^2}{z^2+\gamma^2}\right), \quad z \in \mathbb{R}.$$

Logarithmic Input Constraint

$$\left\{X : \ \mathbb{E}\left[\log\left(\left(\frac{A+\gamma}{A}\right)^2 + \left(\frac{X}{A}\right)^2\right)\right] \le \log(4)\right\}$$

Capacity

$$C(A,\gamma) = \max_X I(X;Y) = \log\left(\frac{A}{\gamma}\right)$$

achieved with $X \sim \mathcal{C}(0, A-\gamma)$

Fahs and I. Abou-Faycal, "A Cauchy input achieves the capacity of a Cauchy channel under a logarithmic constraint," in *Proc. IEEE Int. Symp. Inf. Theory*, Honolulu, HI, 2014, pp. 3077–3081.

Input has infinite power

Can we find a 'good' finite power input?

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Simple Application of OW

Theorem: Let $X_D \sim \mathrm{PAM}(N, P)$ with $N = \lfloor A/\gamma \rfloor$ and

$$\mathbb{E}[X_D^2] = P = A^2 \left(4 - \left(\frac{A+\gamma}{A} \right)^2 \right).$$

Then for $\frac{A}{\gamma} \geq \frac{1}{\sqrt{2}-1}$

$$C(A,\gamma) - I(X_D; Y) \leq 2.5 \,\text{nats}.$$

A. Dytso, M. Goldenbaum*, H. V. Poor*, and S. Shamai, "A Generalized Ozarow-Wyner Capacity Bound with Applications," Submitted to *Proc. IEEE Int. Symp. Inf. Theory*, 2017

Concluding Remarks

- Properties of the MMPE and MMSE
- SNR Evolution of the MMSE of Optimal Codes
- Generalized Ozarow-Wyner Bound
- Approximate Capacity Result for Additive Channel

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Outlook

- Relevant version of the Single-Crossing Point Property for MIMO

Ronit Bustin, Miquel Payaro , Daniel P. Palomar and Shlomo Shamai(Shitz),"On MMSE Properties and I-MMSE Implications in Parallel MIMO GaussianChannels," IEEE Transactions Information Theory, vol. 59, no. 2, pp. 818-844, February 2013.

- Bounds on Condition Reyni Entropy via MMPE
- Applications of OW bound to MIMO and non-Additive Channel
- Entropy Power Inequalities: Extensions and Insights
- Bottleneck Type Problems

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Thank you

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Abstract: This talk will focus on the recent applications of Information-Estimation Relations to Gaussian Networks. In the first part of the talk, we will go over recent connections between estimation theoretic and information theoretic measures. The estimation theoretic measures that would be of importance to us are the Minimum Means p-th Error (MMPE) and its special case the Minimum Mean Square Error (MMSE). As will be demonstrated, the MMSE can be very useful in bounding mutual information via the I-MMSE relationship of Guo-Shamai-Verdu, and the MMPE can be used to bound the conditional entropy via the moment entropy inequality.

In the second part of the talk, we will discuss several applications of Information-Estimation Relations in Gaussian noise networks. As the first application, we show how the I-MMSE relationship can be used to determine the behavior, for every signal-to-noise ratio (SNR), of the mutual information and the MMSE of the transmitted codeword for the setting of the Gaussian Broadcast Channel and the Gaussian Wiretap Channel.

As a second application, the notion of the MMPE is used to generalize the Ozarow-Wyner lower bound on the mutual information for discrete inputs on Gaussian noise channels.

A short outlook of future applications concludes the presentation.
This work is in collaboration with H. Vincent Poor, Ronit Bustin, Daniela Tuninetti, Natasha Devroye and Shlomo Shamai.

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