Cloud Radio Access Networks, Distributed Information Bottleneck, and more: A Unified Information Theoretic View

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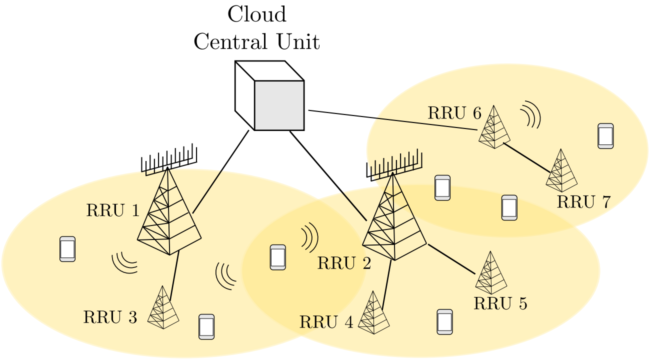
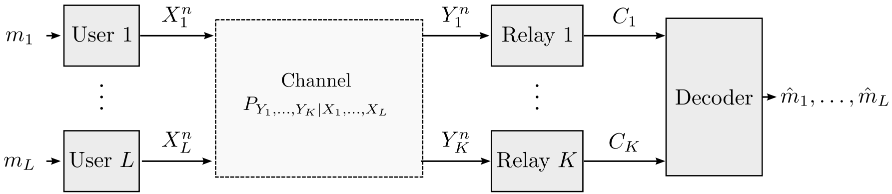
Outline

Introduction



- Cloud radio access network (C-RAN) architecture:
  - Heterogeneous dense networks;
  - Base stations (BSs), macro, pico, femto, operate as radio units (RUs);
  - Baseband processing takes place in the "cloud" or a central unit (CU).

Uplink Cloud RAN



- Multiple access relay channel in which L users communicate with a common destination through K relay nodes.

- Decoder interested in $\hat{m}_1, \dots, \hat{m}_L$ such that, for n large enough,

$$\Pr\{(m_1, \dots, m_L) \neq (\hat{m}_1, \dots, \hat{m}_L)\} \to 0$$

- The capacity region of this model is still to be found
  - problem open even in seemingly simpler cases, e.g., one user and two relays (the diamond channel), parallel Gaussian relay channel *[Schein-Gallager '00]*.

Relay Operations

- Main difficulty is in characterizing the optimal relay operation:
  - **Decode-and-Forward (DF):** *[Cover-ElGamal'97], [Kramer-Gastpar'05] ...*
  - **Compute-and-Forward (CompF):** *[Nazer-Gastpar'11], [Nazer et al'12], [Hong-Caire'13]...*
  - **Compress-and-Forward (CoF):** *[Sanderovich et al'09], [Park et al'13], [Zhou et at'13]...*
  - **Noisy Network Coding (NNC):** *[Lim et al'11]...*
  - **Others:** Amplify and Forward, Partial-Decode-Compress-and-Forward *[Cover-ElGamal'97]*, Compute-Quantize-and-Forward *[Estella-Zaidi'16]*.
- Relaying operations can be divided into:
  - Non-oblivious: relays aware of the users' codebooks (modulation, coding...) at all time, e.g., DF, CompF.
  - Oblivious (or Nomadic): *[Sanderovich et al'08]* relays operate without knowledge of the users' codebooks, e.g., CoF, NNC.
- Oblivious processing motivated mainly by practical constraints.
- Formally, obliviousness of the relays to actual codebooks is modeled through randomized encoding *[Sanderovich et al'08], [Lapidoth-Narayan'98]*.

Randomized Encoding as a Model for Obliviousness

- Encoding function at transmitter

$$\phi^n : [1, |\mathcal{X}|^{n2^{nR}}] \times [1, 2^{nR}] \to \mathcal{X}^n$$

which maps:

  - a codebook index $F \in [1, |\mathcal{X}|^{n2^{nR}}]$ and
  - a message $M \in [1, 2^{nR}]$

into a codeword

$$X^n(F, M) = \phi^n(F, M).$$

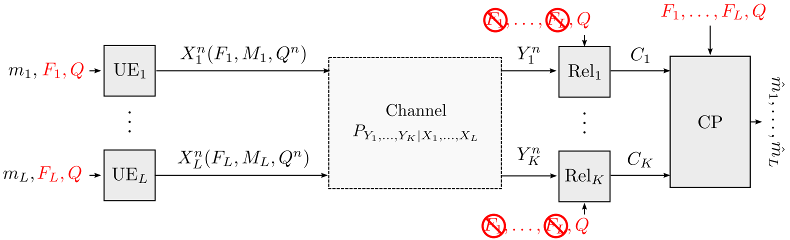
- The pair (p_F, ϕ^n) must satisfy

$$\text{Prob}[X^n(F, M) = x^n] = \prod_{i=1}^{n} p_X(x_i)$$

for some $p_X(x)$, $x \in \mathcal{X}$, where Prob$[\cdot]$ is calculated with respect to

$$p_{F,M}(f, m) = p_F(f) \cdot 2^{-nR}.$$

Oblivious Relay Processing with Enabled Resource-sharing



- Resource-sharing random variable Q^n available at all terminals *[Simeone et al'11]*.
- Q^n way easier to share, (e.g., on/off activity).
- Memoryless Channel: $P_{Y_1,\ldots,Y_K|X_1,\ldots,X_1}$
- User $l \in \{1, \ldots, L\}$: $\phi_l^n : [1, |\mathcal{X}_l|^{n2^{nR_l}}] \times [1, 2^{nR_l}] \times \mathcal{Q}^n \to \mathcal{X}_l^n$
- Relay $k \in \{1, \ldots, K\}$: $g_k^n : {\mathcal{Y}_k}^n \times \mathcal{Q}^n \to [1, 2^{nC_k}]$
- Decoder:

$$\psi^n : [1, |\mathcal{X}_1|^{n2^{nR_1}}] \times \cdots \times [1, 2^{nC_K}] \times \mathcal{Q}^n \to [1, 2^{nR_1}] \times \ldots \times [1, 2^{nR_L}]$$

Main Capacity Results

Single-letter characterizations of:

1) *Capacity Region* of the Class of DM CRAN channels satisfying

$$Y_k^n -\!\!\circ\!\!- X_{\mathcal{L}}^n -\!\!\circ\!\!- Y_{\mathcal{K}\setminus k}^n,$$

2) *Capacity Region of Gaussian MIMO Channels with Gaussian Inputs*
   - In particular, we show that Gaussian auxiliaries are optimal.
   - And, time (frequency) sharing is in general needed.

3) *Inner and Outer Bounds* for General DM Model (i.e., Without the Markov Chain).

Capacity Region of a Class of CRAN Channels

Theorem

For the class of discrete memoryless channels satisfying

$$Y_k -\!\!\circ\!\!- X_{\mathcal{L}} -\!\!\circ\!\!- Y_{\mathcal{K}\setminus k}$$

with oblivious relay processing and enabled resource-sharing, a rate tuple $(R_1, \ldots, R_L)$ *is achievable if and only if for all* $\mathcal{T} \subseteq \mathcal{L}$ *and for all* $\mathcal{S} \subseteq \mathcal{K}$*,*

$$\sum_{t\in\mathcal{T}} R_t \leq \sum_{s\in\mathcal{S}} [C_s - I(Y_s; U_s|X_{\mathcal{L}}, Q)] + I(X_{\mathcal{T}}; U_{\mathcal{S}^c}|X_{\mathcal{T}^c}, Q),$$

for some joint measure of the form

$$P_Q \prod_{l=1}^{L} P_{X_l|Q} \prod_{k=1}^{K} P_{Y_k|X_{\mathcal{L}}} \prod_{k=1}^{K} P_{U_k|Y_k,Q},$$

with the cardinality of Q *bounded as* $|\mathcal{Q}| \leq K + 2$.

Direct Part

Capacity region achievable with

- **Compress-and-Forward with Joint-Decompression-Decoding**
  - Generalization of scheme from *[Sanderovich et al'09]* to L users.
  - Based on compress-and-forward à la Cover-El Gamal with joint decoding and decompression **(JDD)** at the CP.
  - Gaussian inputs are not optimal for finite capacity fronthauls.
- Separate Decompression-Decoding not optimal in general.
- **Noisy Network Coding**
  - Particular case of *[Theorem 1, Lim et al'11]*.

Sum-rate achievable also with

- **Compress-and-Forward with Separate Decompression-Decoding (SDD)**
  - The CP decodes explicitly the compression indices first and then decodes the users' transmitted messages.

Outline of Converse Part

- Define $U_{i,k} := (J_k, Y_k^{i-1})$ and $\bar{Q}_i := (X_{\mathcal{L}}^{i-1}, X_{\mathcal{L},i+1}^n, \tilde{Q})$.

  Fano's Inequality: $H(m_{\mathcal{T}}|J_{\mathcal{K}}, F_{\mathcal{L}}, \tilde{Q}) \leq \epsilon_n$ for $\mathcal{T} \subseteq \mathcal{L}$,

- **Upper bound on entropy term:** For $\mathcal{T} \subseteq \mathcal{L} := \{1, \ldots, L\}$,

$$H(X_{\mathcal{T}}^n|X_{\mathcal{T}^c}^n, J_{\mathcal{K}}, Q^n) \leq \sum_{i=1}^{n} H(X_{\mathcal{T},i}|X_{\mathcal{T}^c,i}, \bar{Q}_i) - n\sum_{t\in\mathcal{T}} R_t := n\Gamma_{\mathcal{T}}$$

  - Follows from

$$\begin{aligned} n\sum_{t\in\mathcal{T}} R_t = H(m_{\mathcal{T}}) &= I(m_{\mathcal{T}}; J_{\mathcal{K}}, F_{\mathcal{L}}, \tilde{Q}) + H(m_{\mathcal{T}}|J_{\mathcal{K}}, F_{\mathcal{L}}, \tilde{Q}) \\ &\leq I(m_{\mathcal{T}}, F_{\mathcal{T}}; J_{\mathcal{K}}|F_{\mathcal{T}^c}, \tilde{Q}) + n\epsilon_n \\ &\leq H(X_{\mathcal{T}}^n|X_{\mathcal{T}^c}^n, \tilde{Q}) - H(X_{\mathcal{T}}^n|X_{\mathcal{T}^c}^n, J_{\mathcal{K}}, \tilde{Q}) + n\epsilon_n \end{aligned}$$

- Reminiscent of log-loss penalty criterion in multi-terminal source coding *[Courtade-Weissman'14]*:

$$H(X^n|J_{\mathcal{K}}) \leq \mathrm{E}[d_{\log}(X^n; \hat{X}^n)] \simeq n(H(X) - I(X; \hat{X}))$$

Outline of Converse Part (Cont.)

- **Bound on users' rates:** For $\mathcal{T} \subseteq \mathcal{L}$

$$n \sum_{t \in \mathcal{T}} R_t \leq I(m_{\mathcal{T}}, F_{\mathcal{T}}; J_{\mathcal{K}} | F_{\mathcal{T}^c}, \tilde{Q}) + n\epsilon_n \leq \sum_{i=1}^{n} I(X_{\mathcal{T},i}; U_{\mathcal{K},i} | X_{\mathcal{T}^c,i}, \bar{Q}_i) + n\epsilon_n$$

- **Bound on relays' rates:** For $\mathcal{S} \subseteq \mathcal{K} := \{1, \dots, K\}$

$$\begin{aligned} n \sum_{k \in \mathcal{S}} C_k \geq \sum_{k \in \mathcal{S}} H(J_k) &\geq I(X_{\mathcal{T}}^n, Y_{\mathcal{S}}^n; J_{\mathcal{S}} | X_{\mathcal{T}^c}^n, J_{\mathcal{S}^c}, \tilde{Q}) \\ &\geq \sum_{i=1}^{n} H(X_{\mathcal{T},i} | X_{\mathcal{T}^c,i}, U_{\mathcal{S}^c,i}, \bar{Q}_i) - n\Gamma_{\mathcal{T}} + I(Y_{\mathcal{S}}^n; J_{\mathcal{S}} | X_{\mathcal{L}}^n, J_{\mathcal{S}^c}, \tilde{Q}) \\ &= nR_{\mathcal{T}} - \sum_{i=1}^{n} I(X_{\mathcal{T},i}; U_{\mathcal{S}^c,i} | X_{\mathcal{T}^c,i}, \bar{Q}_i) + \sum_{k \in \mathcal{S}} \sum_{i=1}^{n} I(Y_{k,i}; U_{k,i} | X_{\mathcal{L},i}, \bar{Q}_i) \end{aligned}$$

where we used the upper bound on the entropy and the Markov chain

$$Y_k -\!\!\circ\!\!- X_{\mathcal{L}} -\!\!\circ\!\!- Y_{\mathcal{K} \setminus k}$$

Remarks

- Sum-rate achievable with CF with JDD given by

$$R_{\text{JDD}}^{\text{sum}} = \max \min_{\mathcal{S} \subseteq \mathcal{K}} \Big\{ \sum_{s \in \mathcal{S}} C_s - I(Y_S; U_S | X_{\mathcal{L}}, U_{S^c}, Q) + I(X_{\mathcal{L}}; U_{\mathcal{S}^c} | Q) \Big\}.$$

- Using properties of sub-modular functions, we show that CF with SDD (and even the low-complexity version of it, consisting in *sequential* decompression followed by *sequential* decoding, denoted as SWZ) achieve the same sum-rate as CF with JDD.
- Note, however, that time-sharing is generally needed for the three to achieve optimal sum-rate!

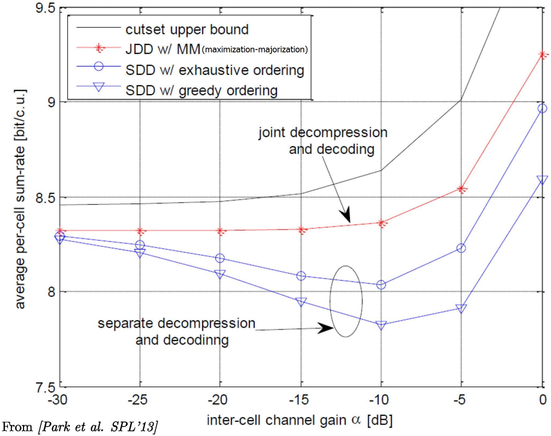
Theorem

For any $P_{Y_1,\ldots,Y_K|X_1,\ldots,X_L}$*, not necessarily satisfying* $Y_k -\!\!\circ\!\!- X_{\mathcal{L}} -\!\!\circ\!\!- Y_{\mathcal{K}\setminus k}$*, we have*

$$R_{\text{JDD}}^{\text{sum}} = R_{\text{SDD}}^{\text{sum}} = R_{\text{SWZ}}^{\text{sum}}$$

- In particular, for MIMO Gaussian channels recovers *[Zhou et al.'16]*.
- In terms of rate-region, CF with JDD generally outperforms CF with SDD.

Numerical example: 3 Cell Uplink Circular Wyner Model



From *[Park et al. SPL'13]*

- For SWZ, optimizing over relay ordering improves performance, in general
- Without time (or resource)-sharing, as is in the figure, SDD may achieve smaller sum-rate than JDD.

Memoryless MIMO Gaussian Model

- The channel output at relay node k with M_k antennas:

$$\mathbf{Y}_k = \mathbf{H}_{k,\mathcal{L}} \; [\mathbf{X}_1^T, \ldots, \mathbf{X}_L^T]^T + \mathbf{N}_k,$$

  where

  - User l with N_l antennas transmits $\mathbf{X}_l$ with $\mathrm{E}[\|\mathbf{X}_l\|^2] \preceq \mathbf{K}_l$.
  - Relay k with M_k antennas.
  - $\mathbf{H}_{k,\mathcal{L}} = [\mathbf{H}_{k,1}, \ldots, \mathbf{H}_{k,L}]$, $\mathbf{H}_{k,l}$ channel between user l and relay k.
  - $\mathbf{N}_k \sim \mathcal{CN}(0, \mathbf{\Sigma}_k)$ is AWGN noise at relay k, assumed independent.

- Outputs satisfy $Y_k -\!\!\circ\!\!- X_{\mathcal{L}} -\!\!\circ\!\!- Y_{\mathcal{K}\setminus k}$.

- Theorem 1 characterizes its **capacity region**. Finding the optimal $U_1, \ldots, U_K$ is generally not easy.

Capacity under Gaussian Signaling and Enabled Resource-Sharing

Theorem (Capacity Region under Gaussian Input with Enabled Resource-Sharing)

Let the input vectors use Gaussian Signaling with Enabled Resource-Sharing, i.e.,

$$\mathbf{X}_{l,q} \sim \mathcal{CN}(0, \mathbf{K}_{l,q}) \quad q \in \{1, \dots, |\mathcal{Q}|\} \qquad \sum_{q \in \mathcal{Q}} p_Q(q) \mathbf{K}_{l,q} \leq \mathbf{K}_l$$

The capacity region is given by the set of all rate tuples $(R_1, \dots, R_L)$ *satisfying that for all* $\mathcal{T} \subseteq \mathcal{L}$ *and all* $\mathcal{S} \subseteq \mathcal{K}$

$$\sum_{t \in \mathcal{T}} R_t \leq \sum_{k \in \mathcal{S}} \left[C_k - \mathrm{E}_Q \log \frac{|\boldsymbol{\Sigma}_k^{-1}|}{|\boldsymbol{\Sigma}_k^{-1} - \mathbf{B}_{k,q}|} \right] + \mathrm{E}_Q \log \frac{|\sum_{k \in \mathcal{S}^c} \mathbf{H}_{k,\mathcal{T}}^H \mathbf{B}_{k,q} \mathbf{H}_{k,\mathcal{T}} + \mathbf{K}_{\mathcal{T},}^{-1}|}{|\mathbf{K}_{\mathcal{T},q}^{-1}|},$$

for some $\mathbf{0} \preceq \mathbf{B}_{k,q} \preceq \boldsymbol{\Sigma}_k^{-1}$, *where* $\mathbf{H}_{k,\mathcal{T}}$ *is the channel between* $\mathbf{X}_{\mathcal{T}}$ *and* $\mathbf{Y}_k$.

- Extends *[Theorem 5, Sanderovich et al'09]* to L users and MIMO.
- Achievable with $\mathbf{U}_{k,q} = \mathbf{Y}_{k,q} + \mathbf{Z}_{k,q}$, $\mathbf{Z}_{k,q} \sim \mathcal{CN}(\mathbf{0}, \mathbf{B}_{k,q}^{-1} - \boldsymbol{\Sigma}_{k,q})$, $q \in \mathcal{Q}$.
- Gaussian signaling can be strictly suboptimal *[Sanderovich et al'09]*.

Converse Part

- For (X, U) arbitrarily correlated,

$$\log |(\pi e)\ \mathbf{J}^{-1}(X|U)| \leq h(X|U) \leq \log |(\pi e)\ \mathrm{mmse}(X|U)|$$

- For each $Q = q$,

$$I(\mathbf{Y}_k; \mathbf{U}_k | \mathbf{X}_{\mathcal{L}}, Q = q) = \log |(\pi e)\boldsymbol{\Sigma}_k| - h(\mathbf{Y}_s | \mathbf{X}_{\mathcal{L},q}, \mathbf{U}_{s,q}, Q = q)$$
$$\geq \log \frac{|\boldsymbol{\Sigma}_k^{-1}|}{|\boldsymbol{\Sigma}_k^{-1} - \mathbf{B}_{k,q}|},$$

where $\mathbf{0} \preceq \mathbf{B}_{k,q} \preceq \boldsymbol{\Sigma}_k^{-1}$ is chosen such that

$$\mathrm{mmse}(\mathbf{Y}_k | \mathbf{X}_{\mathcal{L},q}, \mathbf{U}_{k,q}) = \boldsymbol{\Sigma}_k - \boldsymbol{\Sigma}_k \mathbf{B}_{k,q} \boldsymbol{\Sigma}_k$$

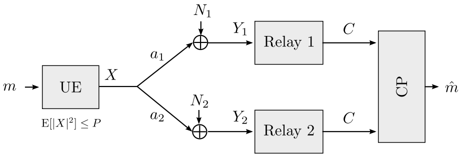
- Also,

$$I(\mathbf{X}_{\mathcal{T}}; \mathbf{U}_{S^c} | \mathbf{X}_{\mathcal{T}^c}, Q = q) = h(\mathbf{X}_{\mathcal{T},q} | \mathbf{X}_{\mathcal{T}^c}, q) - h(\mathbf{X}_{\mathcal{T}} | \mathbf{X}_{\mathcal{T}^c}, \mathbf{U}_{S^c,q}, Q = q)$$
$$\leq \log |\mathbf{K}_{\mathcal{T},q}| + \log \left| \sum_{k \in \mathcal{S}^c} \mathbf{H}_{k,\mathcal{T}}^H \mathbf{B}_{k,q} \mathbf{H}_{k,\mathcal{T}} + \mathbf{K}_{\mathcal{T},q}^{-1} \right|$$

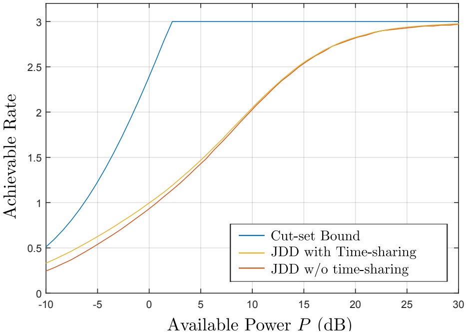
by deBrujin Identity *[Palomar-Verdu'06],[Ekrem-Ulukuss'14], [Zhou et al'17]*

$$\mathbf{J}(\mathbf{X}_{\mathcal{T},q} | \mathbf{X}_{\mathcal{T}^c,q}, \mathbf{U}_{S^c,q}) = \sum_{k \in \mathcal{S}^c} \mathbf{H}_{k,\mathcal{T}}^H \mathbf{B}_{k,q} \mathbf{H}_{k,\mathcal{T}} + \mathbf{K}_{\mathcal{T},q}^{-1}.$$

Resource-sharing Enlarges Capacity Region



$Y_k = a_k X + N_k$, with $\mathrm{E}[X^2] \leq P$ and $N_k \sim \mathcal{N}(0,1)$, $k = 1, 2$



- JDD without resource (time)-sharing, i.e., $Q = \emptyset$
- JDD with resource (time)-sharing ($|\mathcal{Q}| = 2$. Recall that $|\mathcal{Q}| \leq K + 2 = 4$ here)
  - Phase I: UE transmits at P/α for αn samples. Relays compress at C_k/α.
  - Phase II: UE and Relays remain inactive for $(1-\alpha)n$ remaining samples.
- Intuition: For small P, the observations at the relays are too noisy; and, so, it is more advantageous to increase power and compression rate during shorter time.

Capacity under Constant Gaussian Signaling

Theorem (Capacity Region under Constant Gaussian Input)

If the input vectors use constant Gaussian Signaling, i.e.,

$$\mathbf{K}_{1,l} = \cdots = \mathbf{K}_{|\mathcal{Q}|,l} = \mathbf{K}_l, \qquad \mathbf{X}_l \sim \mathcal{CN}(0, \mathbf{K}_l),$$

the capacity region is given by the set of all rate tuples $(R_1, \dots, R_L)$ *satisfying that for all* $\mathcal{T} \subseteq \mathcal{L}$ *and all* $\mathcal{S} \subseteq \mathcal{K}$

$$\sum_{t \in \mathcal{T}} R_t \leq \sum_{k \in \mathcal{S}} \left[C_k - \log \frac{|\mathbf{\Sigma}_k^{-1}|}{|\mathbf{\Sigma}_k^{-1} - \mathbf{B}_k|} \right] + \log \frac{|\sum_{k \in \mathcal{S}^c} \mathbf{H}_{k,\mathcal{T}}^H \mathbf{B}_k \mathbf{H}_{k,\mathcal{T}} + \mathbf{K}_{\mathcal{T}}^{-1}|}{|\mathbf{K}_{\mathcal{T}}^{-1}|},$$

for some $\mathbf{0} \preceq \mathbf{B}_k \preceq \mathbf{\Sigma}_k^{-1}$, *where* $\mathbf{H}_{k,\mathcal{T}}$ *is the channel between* $\mathbf{X}_{\mathcal{T}}$ *and* $\mathbf{Y}_k$.

- Resource-sharing at the relays does not enlarge the capacity region under constant Gaussian Signaling.
- Proof follows from Jensen's Inequality and concavity of log-det.

Capacity under Gaussian Signaling in the High SNR Regime

- High SNR regime model:

$$\mathbf{\Sigma}_k = \epsilon\tilde{\mathbf{\Sigma}}_k; \quad \text{for some } \tilde{\mathbf{\Sigma}}_k \succeq \mathbf{0}, \quad \text{and } \epsilon \to 0.$$

- We have $\mathcal{R}_{\mathrm{GNS}}(C_{\mathcal{K}}) \subset \mathcal{R}_{\mathrm{GTS}}(C_{\mathcal{K}})$, where:
  - $\mathcal{R}_{\mathrm{GTS}}$: Capacity under Gaussian Input with enabled resource-sharing.
  - $\mathcal{R}_{\mathrm{GNS}}$: Capacity under Gaussian Input without resource-sharing ($Q = \emptyset$).

Theorem (Capacity Region under Gaussian Input in High SNR)

If $(R_1, \ldots, R_L) \in \mathcal{R}_{\mathrm{GTS}}(C_{\mathcal{K}})$, *then for any* $\epsilon > 0$, *for some* $\Delta_\epsilon \geq 0$,

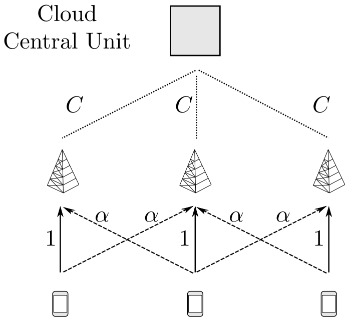
$$(R_1 - \Delta_\epsilon, \ldots, R_L - \Delta_\epsilon) \in \mathcal{R}_{\mathrm{GNS}}(C_{\mathcal{K}})$$

In addition, $\mathcal{R}_{\mathrm{GNS}}(C_{\mathcal{K}}) = \mathcal{R}_{\mathrm{GTS}}(C_{\mathcal{K}})$ *as* $\epsilon \to 0$, *since*

$$\lim_{\epsilon \to 0} \Delta_\epsilon = 0.$$

- For large SNR, the gains due to resource-sharing become limited.

Numerical example: Circular Wyner Model



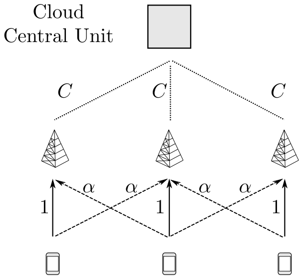
- Each cell contains a single-antenna and a single-antenna RU.
- Inter-cell interference takes place only between adjacent cells (circular).

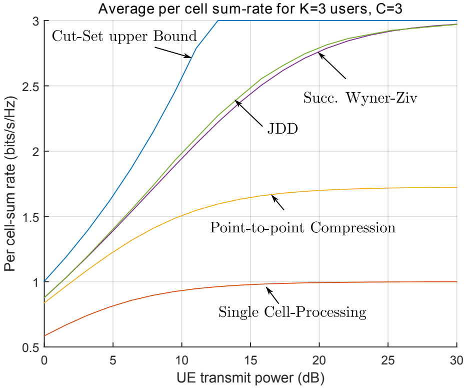
$$Y_k = \alpha X_{k-1} + X_k + \alpha X_{k+1} + N_k$$

where $N_k \sim \mathcal{CN}(0, 1)$

- All RRUs have a fronthaul capacity of C.

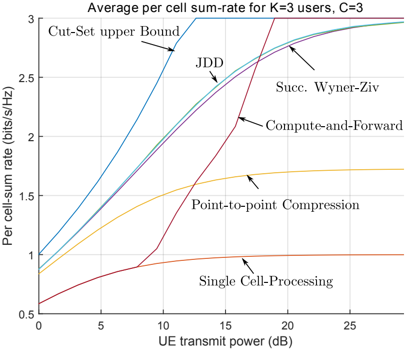
Numerical example: Circular Wyner Model (cont'd)

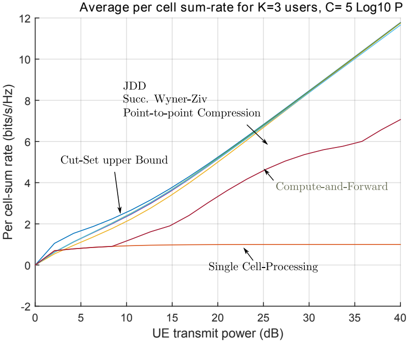




- JDD is capacity achieving under oblivious processing.
- For this simple network, JDD does not provide much gain compared to SDD and SWZ.
  - Here, the schemes SDD and SWZ do not employ resource-sharing.

Cost of Obliviousness





- **Optimal degrees-of-freedom:** when fronthaul capacity grows with SNR, e.g., $C = 5\log_{10}(\mathrm{snr})$. *[Sanderhovich et al'09].*
- Capacity under Gaussian signaling to within a **constant gap** of cut-set bound.
  - If $(R_1, \ldots, R_L)$ is within the cut-set bound, then

$$((R_1-\Delta)^+, \ldots, (R_L-\Delta)^+), \qquad \Delta \leq \begin{cases} \frac{N}{2}(2.45 + \log(\frac{KM}{N})) & \text{for } KM > 2N, \\ \frac{KM+N}{2} & \text{for } KM \leq 2N \end{cases}$$

Inner and Outer Bounds for General CRAN Models

Theorem (Bounds)

For general DM CRAN channels with oblivious relay processing and enabled resource-sharing, a rate tuple $(R_1, \dots, R_L)$ is achievable if (only if) for all $\mathcal{T} \subseteq \mathcal{L}$ and for all $\mathcal{S} \subseteq \mathcal{K}$,

$$\sum_{t \in \mathcal{T}} R_t \leq \sum_{s \in \mathcal{S}} C_s - I(Y_S; U_S | X_{\mathcal{L}}, U_{S^c}, Q) + I(X_{\mathcal{T}}; U_{\mathcal{S}^c} | X_{\mathcal{T}^c}, Q),$$

Inner bound: *for some pmf* $P_Q \prod_{l=1}^{L} P_{X_l|Q} P_{Y_{\mathcal{K}}|X_{\mathcal{L}}} \prod_{k=1}^{K} P_{U_k|Y_k,Q}$.

Outer bound: *for some* $(Q, X_{\mathcal{L}}, Y_{\mathcal{K}}, U_{\mathcal{K}}, W)$

- *distributed according to* $P_Q \prod_{l=1}^{L} P_{X_l|Q} \; P_{Y_{\mathcal{K}}|X_{\mathcal{L}}} \; P_{W|Q}$
- $u_k = f_k(w, y_k, q)$ *for some random variable* W *and some deterministic functions* $\{f_k\}$, $k \in \mathcal{K}$.

- Problem is challenging, as it includes Korner-Marton modulo-sum problem *[Korner-Marton'79]* as a special case.

Information Bottleneck

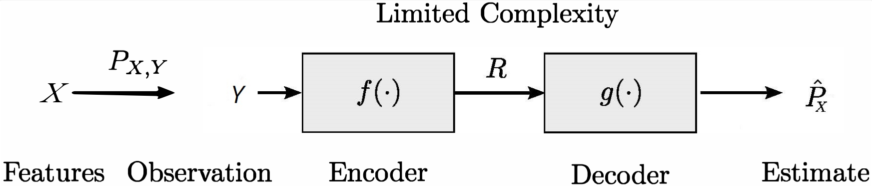
$$X \longrightarrow Y \longrightarrow U$$

- Efficiency of a given representation $U = f(Y)$ measured by the pair

  Rate (or *Complexity*): $I(U;Y)$ and Information (or *Relevance*): $I(U;X)$

- Information $I(X;U)$ can be achieved by OBLIVIOUS coding Y while with the logarithmic distortion with respect to X
- Single letter-wise, U is not necessarily a deterministic function of Y
- The non-oblivious bottleneck problem is immediate as the $\min(I(X;Y), R)$ is achievable by having the relay decoding the message transmitted by X
- The bottleneck problem connects to many timely aspects, such as 'deep learning' *[Tishby-Zaslavsky, ITW'15]*.

Digression: Learning via the Information Bottleneck Method



- Preserving all the information about X that is contained in Y, i.e., $I(X;Y)$, requires high *complexity* (in terms of *minimum description coding length*).
  - Other measures of complexity may be (Vapnik-Chervonenkis) VC-dimension, covering numbers, ..
- Efficiency of a given representation $U = f(Y)$ measured by the pair

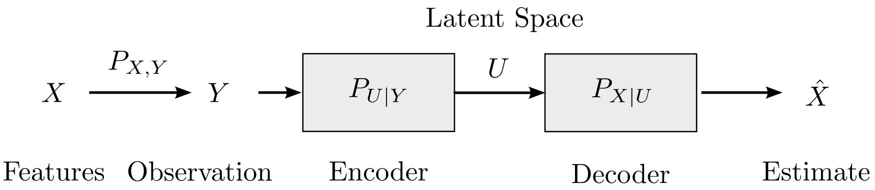
  Complexity: $I(U;Y)$ and Relevance: $I(U;X)$

- Example:

$$\max_{p(u|x)} I(U;X) \quad \text{s.t.} \quad I(U;Y) \leq R, \quad \text{for} \quad 0 \leq R \leq H(Y)$$

$$\min_{p(u|x)} I(U;Y) \quad \text{s.t.} \quad I(U;X) \geq \Delta, \quad \text{for} \quad 0 \leq \Delta \leq I(X;Y)$$

Basically, a Remote Source Coding Problem !



- Reconstruction at decoder is under log-loss measure,

$$R(\Delta) = \min_{p(u|y)} I(U;Y)$$

where the minimization is over all conditional pmfs $p(u|y)$ such that

$$\mathbb{E}[\ell_{\mathsf{log}}(X,U)] \leq H(X) - H(X|U) = H(X) - \Delta$$

- R. L. Dobrushin and B. S. Tsybakov, *"Information transmission with additional noise"*, IRE Tran. Info. Theory, Vol. IT-8, pp. 293-304, 1962.

- H. Witsenhausen, A. Wyner, *"A conditional entropy bound for a pair of discrete random variables"*, IEEE Trans. on Info. Theory, Vol. 21, pp. 493-501, 1975.

- Solution also coined as the Information Bottleneck Method *[Tishby'99]*

$$L_{\mathrm{IB}}(\beta, P_{X,Y}) = \min_{p(u|y)} I(Y;U) - \beta I(X;U)$$

Other Connections

- **Common Reconstruction**. Because $U \multimap Y \multimap X$, we have

$$\begin{aligned} I(U;X) &= I(U;Y) - I(U;Y|X) \\ &\leq R - I(U;Y|X) \end{aligned}$$

- Y. Steinberg, *"Coding and common reconstruction"*, IEEE Trans. on Info. Theory, vol. 55, no. 11, pp. 4995–5010 (X – side information is not used for the 'source' Y common reconstruction).

- **Information Combining**

$$I(Y;U,X) = I(U;Y) + I(X;Y) - I(U;X) \qquad (\text{since } U \multimap Y \multimap X)$$

Since $I(X;Y)$ is given and $I(Y;U) = R$, maximizing $I(U;X)$ is equivalent to minimizing $I(Y;U,X)$.

- I. Sutskover, S. Shamai and J. Ziv, "Extremes of Information Combining", IEEE Trans. Inform. Theory, vol. 51, no. 4, pp. 1313–1325, April 2005.

- I. Land and J. Huber, *"Information combining,"* Foundations and trends in Commun. and Inform. Theory, vol. 3, pp. 227–330, Nov. 2006.

Other Connections (Cont.)

- **Wyner-Ahlswede-Körner Problem**

  If X and Y are encoded at rates R_X and R_Y, respectively. For given $R_Y = R$, the minimum rate R_X that is needed to recover X losslessly is

$$R_X^\star(R) = \min_{p(u|y)\,:\,I(U;Y)\,\leq\,R} H(X|U)$$

So, we get

$$\max_{p(u|y)\,:\,I(U;Y)\leq R} I(U;X) = H(X) - R_X^\star(R)$$

- R. F. Ahlswede and J. Korner, *"Source coding with side information and a converse for degraded broadcast channels"*, IEEE Trans. on Info. Theory, Vol. 21, pp. 629-637, 1975.

- A. D. Wyner, *"On source coding with side information at the decoder"*, IEEE Trans. on Info. Theory, Vol. 21, pp. 294-300, 1975.

Vector Gaussian Information Bottleneck

- $(\mathbf{X}, \mathbf{Y})$ jointly Gaussian, $\mathbf{X} \in \mathbb{R}^N$ and $\mathbf{Y} \in \mathbb{R}^M$
- Optimal encoding $P_{U|Y}$ is a noisy linear projection to a subspace whose dimensionality is determined by the bottleneck Lagrangian multiplier β *[Chechik et al. '05]*

$$\mathbf{U} = \mathbf{A}\mathbf{Y} + \mathbf{Z}, \qquad \mathbf{Z} \sim \mathcal{N}(\mathbf{0}, \mathbf{I})$$

where

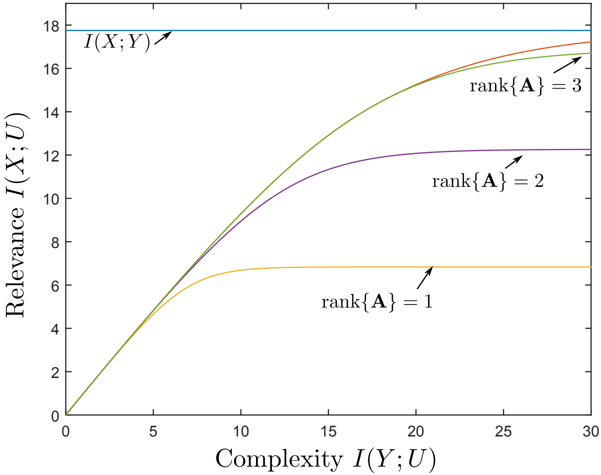
$$\mathbf{A} = \begin{cases} [\mathbf{0}^T; \ldots; \mathbf{0}^T], & \text{if } 0 \leq \beta \leq \beta_1^c \\ [\alpha_1 \mathbf{v}_1^T; \mathbf{0}^T; \ldots; \mathbf{0}^T], & \text{if } \beta_1^c \leq \beta \leq \beta_2^c \\ [\alpha_1 \mathbf{v}_1^T; \alpha_2 \mathbf{v}_2^T; \mathbf{0}^T; \ldots; \mathbf{0}^T], & \text{if } \beta_2^c \leq \beta \leq \beta_3^c \\ \qquad \vdots & \end{cases}$$

and $\{\mathbf{v}_1^T, \ldots, \mathbf{v}_N^T\}$ are the left eigenvectors of $\mathbf{\Sigma}_{y|x}\mathbf{\Sigma}_y^{-1}$, sorted by their ascending eigenvalues $\{\lambda_1, \ldots, \lambda_N\}$; $\beta_i^c = 1/(1-\lambda_i)$ are critical β values; $r_i = \mathbf{v}_i^T \mathbf{\Sigma}_y \mathbf{v}_i$ and

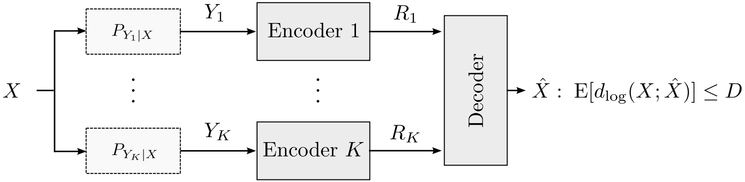
$$\alpha_i = \sqrt{\frac{\beta(1-\lambda_i) - 1}{\lambda_i r_i}}$$

Rate-Information Trade-off Gaussian Vector Channel [Winkelbauer-Matz, ISIT'14].

Rate-Information Curve



CEO Source Coding Problem under Log-Loss



- CEO source coding problem under log-loss distortion:

$$d_{\log}(x,\hat{x}) := \log\left(\frac{1}{\hat{x}(x)}\right)$$

  where $\hat{x} \in \mathcal{P}(\mathcal{X})$ is a probability distribution on $\mathcal{X}$.

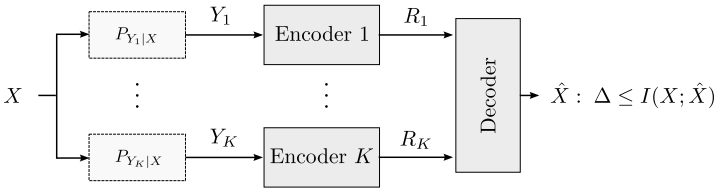
- Characterization of rate-distortion region in *[Courtade-Weissman'14]*
  - Key step: log-loss admits a lower bound in the form of conditional entropy of the source conditioned on the compression indices:

$$nD \geq \mathrm{E}[d_{\log}(X^n;\hat{X}^n)] \geq H(X^n|J_{\mathcal{K}}) = H(X^n) - I(X^n;J_{\mathcal{K}})$$

- Converse of Theorem 1 for Oblivious CRAN leverages on this relation applied to multiple channel inputs, which can be designed.

  Multiple description CEO problem-logloss distortion (Pichler-Piantanida-Matz, ISIT'17].

Distributed Information Bottleneck



- Information Bottleneck introduced by *[Tishby'99]* and *[Witsenhausen'80]*
- It is a CEO source-coding problem under log-loss!

Theorem (Distributed Information Bottleneck [Estella-Zaidi, IZS'18])

The D-IB region is the set of all tuples $(\Delta, R_1, \ldots, R_K)$ *which satisfy*

$$\Delta \leq \sum_{k \in \mathcal{S}} [R_k - I(Y_k; U_k | X, Q)] + I(X; U_{\mathcal{S}^c} | Q), \qquad \text{for all } \mathcal{S} \subseteq \mathcal{K}$$

for some joint pmf $p(q)p(x) \prod_{k=1}^{K} p(y_k|x) \prod_{k=1}^{K} p(u_k|y_k, q)$.

Vector Gaussian Distributed Information Bottleneck

- $(\mathbf{Y}_1, \cdots, \mathbf{Y}_K, \mathbf{X})$ jointly Gaussian, $\mathbf{Y}_k \in \mathbb{R}^N$ and $\mathbf{X} \in \mathbb{R}^M$,

$$\mathbf{Y}_k = \mathbf{H}_k \mathbf{X} + \mathbf{N}_k, \qquad \mathbf{N}_k \sim \mathcal{N}(\mathbf{0}, \mathbf{\Sigma}_{\mathbf{n}_k})$$

- Optimal encoding $P^*_{U_k|Y_k}$ is Gaussian and $Q = \emptyset$ *[Estella-Zaidi'17]*

Theorem (Estella-Zaidi, IZS'18)

If $(\mathbf{X}, \mathbf{Y}_1, \ldots, \mathbf{Y}_K)$ are jointly Gaussian, the D-IB region is given by the set of all tuples $(\Delta, R_1, \ldots, R_L)$ satisfying that for all $\mathcal{S} \subseteq \mathcal{K}$

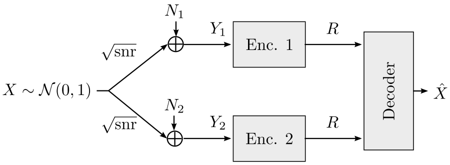
$$\Delta \leq \sum_{k \in \mathcal{S}} [R_k + \log |\mathbf{I} - \mathbf{B}_k|] + \log \left| \sum_{k \in \mathcal{S}^c} \bar{\mathbf{H}}_k^H \mathbf{B}_k \bar{\mathbf{H}}_k + \mathbf{I} \right|$$

for some $\mathbf{0} \preceq \mathbf{B}_k \preceq \mathbf{I}$, where $\bar{\mathbf{H}}_k = \mathbf{\Sigma}_{\mathbf{n}_k}^{-1/2} \mathbf{H}_k \mathbf{\Sigma}_{\mathbf{x}}^{1/2}$, and achievable with

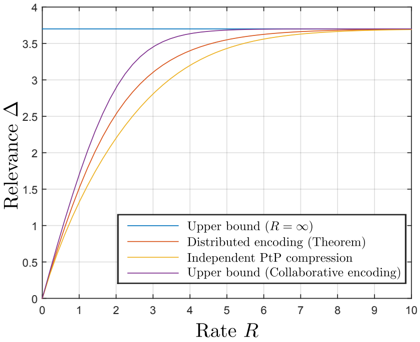
$$p^*(\mathbf{u}_k | \mathbf{y}_k, q) = \mathcal{CN}(\mathbf{y}_k, \Sigma_{\mathbf{n}_k}^{1/2} (\mathbf{B}_k - \mathbf{I}) \mathbf{\Sigma}_{\mathbf{n}_k}^{1/2})$$

- Reminiscent of the sum-capacity in Gaussian Oblivious CRAN with Constant Gaussian Input constraint.

Example



$Y_k = \sqrt{\mathrm{snr}} X + N_k$, with $N_k \sim \mathcal{N}(0,1)$, $k = 1, 2$



- Optimal information (relevance):

$$\Delta^*(R, \mathrm{snr}) = \log_2\left(1 + 2\,\mathrm{snr}\,2^{-2R}\left(2^{2R} + \mathrm{snr} - \sqrt{\mathrm{snr}^2 + (1 + 2\,\mathrm{snr})\,2^{2R}}\right)\right)$$

- Collaborative encoding upper bound: (Y_1, Y_2) encoded at rate $2R$

$$\Delta_{\mathrm{ub}}(R, \mathrm{sr}) = \log_2(1 + 2\,\mathrm{snr}) - \log_2(1 + 2\,\mathrm{snr}\,2^{-2R})$$

- Lower bound: Y_1 and Y_2 independently encoded

$$\Delta_i(R, \mathrm{snr}) = \log_2(1 + 2\,\mathrm{snr} - \mathrm{snr}\,2^{-R}) - \log_2(1 + \mathrm{snr}\,2^{-R})$$

The Distributed Information Bottleneck for Learning

- For simplicity, we look at the D-IB under sum-rate *[Estella-Zaidi'17]*

$$P^*_{U_k|Y_k} = \arg \min_{P_{U_k|Y_k}} I(X;U_{\mathcal{K}}) + \beta \sum_{k=1}^{K} [I(Y_k;U_k) - I(X;U_k)]$$

- The optimal encoders-decoder of the D-IB under sum-rate constraint satisfy the following self consistent equations,

$$p(u_k|y_k) = \frac{p(u_k)}{Z(\beta,u_k)} \exp\left(-\psi_s(u_k,y_k)\right),$$

$$p(x|u_k) = \sum_{y_k \in \mathcal{Y}_k} p(y_k|u_k)p(x|y_k)$$

$$p(x|u_1,\ldots,u_K) = \sum_{y_{\mathcal{K}} \in \mathcal{Y}_{\mathcal{K}}} p(y_{\mathcal{K}})p(u_{\mathcal{K}}|y_{\mathcal{K}})p(x|y_{\mathcal{K}})/p(u_{\mathcal{K}})$$

where

$$\psi_s(u_k,y_k) := D_{\mathrm{KL}}(P_{X|y_k}||Q_{X|u_k}) + \frac{1}{s}\mathrm{E}_{U_{\mathcal{K}\setminus k}|y_k}[D_{\mathrm{KL}}(P_{X|U_{\mathcal{K}\setminus k},y_k}||Q_{X|U_{\mathcal{K}\setminus k},u_k}))].$$

- Alternating iterations of these equations converge to a a solution for any initial $p(u_k|x_k)$, similarly to a Blahut-Arimoto algorithm.

D-IB for Vector Gaussian Sources: Iterative Optimization

- $(\mathbf{Y}_1, \cdots, \mathbf{Y}_K, \mathbf{X})$ jointly Gaussian, $\mathbf{Y}_k \in \mathbb{R}^N$ and $\mathbf{X} \in \mathbb{R}^M$,

$$\mathbf{Y}_k = \mathbf{H}_k \mathbf{X} + \mathbf{N}_k, \qquad \mathbf{N}_k \sim \mathcal{N}(\mathbf{0}, \mathbf{I})$$

- Optimal encoding $P^*_{U_k|Y_k}$ is Gaussian *[Estella-Zaidi'17]* and given by

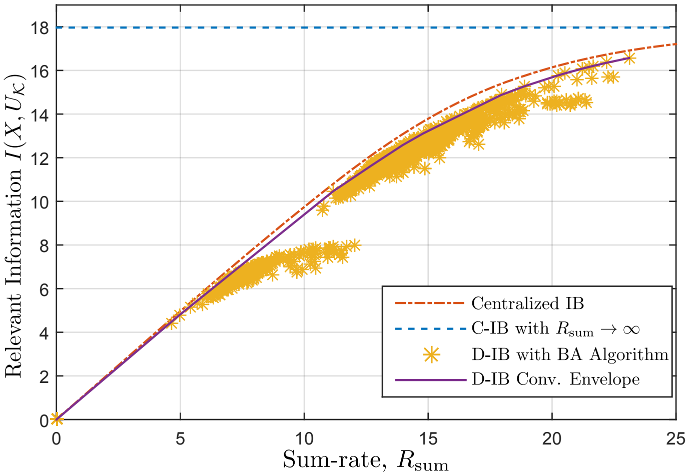
$$\mathbf{U}_k = \mathbf{A}_k \mathbf{Y}_k + \mathbf{Z}_k, \qquad \mathbf{Z}_k \sim \mathcal{N}(\mathbf{0}, \mathbf{\Sigma}_{z,k})$$

- For this class of distributions, the updates in the Blahut-Arimoto type algorithm simplify to:

$$\mathbf{\Sigma}_{\mathbf{z}_k^{t+1}} = \left(\left(1 + \frac{1}{\beta}\right) \mathbf{\Sigma}^{-1}_{\mathbf{u}_k^t|\mathbf{x}} - \frac{1}{s} \mathbf{\Sigma}^{-1}_{\mathbf{u}_k^t|\mathbf{u}^t_{\mathcal{K}\setminus k}} \right)^{-1},$$

$$\mathbf{A}_k^{t+1} = \mathbf{\Sigma}^{-1}_{\mathbf{z}_k^{t+1}} \left(\left(1 + \frac{1}{\beta}\right) \mathbf{\Sigma}^{-1}_{\mathbf{u}_k^t|\mathbf{x}} \mathbf{A}_k^t (\mathbf{I} - \mathbf{\Sigma}_{\mathbf{y}_k|\mathbf{x}} \mathbf{\Sigma}^{-1}_{\mathbf{y}_k}) - \frac{1}{\beta} \mathbf{\Sigma}^{-1}_{\mathbf{u}_k^t|\mathbf{u}^t_{\mathcal{K}\setminus k}} \mathbf{A}_k^t (\mathbf{I} - \mathbf{\Sigma}_{\mathbf{y}_k|\mathbf{u}^t_{\mathcal{K}\setminus k}} \mathbf{\Sigma}^{-1}_{\mathbf{y}_k}) \right).$$

D-IB for Vector Gaussian Sources (cont'd)



- Performance of distributed-IB is close to that of centralized IB

Wrap Up

- We have studied transmission over a CRAN under oblivious processing constraints at the relays and enabled resource-sharing.
  - i.e., relays are not allowed to know or acquire the users' codebooks.
- Our results shed light on the optimal relay operations:
  - NNC and CF with JDD optimal when the outputs at the relay nodes are conditionally independent on the users inputs.
  - Computed the Capacity Region under Gaussian Inputs in MIMO CRAN.
- Oblivious processing relevant from a practical viewpoint:
  - Bounded rate loss in comparison with the non-oblivious setting.
- Discussed relevant connections with CEO under logarithmic loss and Information Bottleneck Method.

Short Outlook

- Duality issues:
  - Downlink/uplink, e.g., Compute-forward v.s. reverse Compute-forward
  - Gaussian MAC/BC duality extends also for finite-capacity fronthauls $\{C_k\}$

    - See, e.g., Liang Liu, Pratik Patil, and Wei Yu, *"An Uplink-Downlink Duality for Cloud Radio Access Network"*, ISIT'2016 and "Channel Diagonalization for Cloud Radio Access,"to appear `http://arXiv:1802.01807`
  - Duality aspects via information bottleneck interpretations.

- Optimal input distributions under rate-constrained compression at relays.
  - Discrete signaling is already known to sometimes outperform Gaussian signaling for single-user Gaussian CRAN *[Sanderovich et al. '08]*.
  - It is conjectured that the optimal input distribution is discrete.
  - Improved upper bounds (over cut-set) to better evaluate the cost of oblivious processing (á la: Vu-Barnes-Ozgur, arXiv:1701.02043 Gaussian primitive relay).

Short Outlook cont.'

- Bounds on general bottleneck problems (Painsky-Tishby, arXiv:1711.02421).
- A variety of related C-RAN & Distributed bottleneck problems:
  - Impact of block length n [C may not scale linearly with $n \Rightarrow$ courtade conjecture $(C = 1)$ [Courtade-Kumar, IT'14], Huleihel-Ordentlich solution $(n - 1$, arXiv:1701.03119).
  - Bandlimited time-continuous models (Homri-Peleg-Shamai, arXiv:1510.08202).
  - Multi-layer Information Bottleneck Problem (Yang-Piantanida-Gündüz, arXiv:1711.05102).
  - Distributed Information-Theoretic Clustering (Pichler-Piantanida-Matz, arXiv:1602.04605, Dictator Functions, arXiv:1604.02109).

Short Outlook cont.'

- Entropy constraint bottleneck:

  $X - Y - U$

  $\max I(X;U)$ under the constraint $H(U) \leq C$

  practical applications: LZ distortionless compression.

  $\Rightarrow U = f(y)$ a deterministic function Homri-Peleg-Shamai (Oblivious Processing in a Fronthaul Constrained Gaussian Channel, arXiv:1510.08202).

  - The deterministic bottleneck: advantages in complexity as compared to a classical bottleneck: [Strouse-Schwab, arXiv:1604.00268].

Some Related Tutorials

- S.-H. Park, O. Simeone, O. Sahin and S. Shamai (Shitz), “Fronthaul compression for cloud radio access networks,” IEEE Sig. Proc. Mag., Special Issue on Signal Processing for the 5G Revolution, vol. 31, no. 6, pp. 69-79, Nov. 2014.
- M. Peng, C. Wang, V. Lau and H. V. Poor, “Fronthaul-Constrained Cloud Radio Access Networks: Insight and Challenges,” IEEE Wireless Comm., vol. 22, no. 2, pp. 152-160, Apr. 2015.
- Yuhan Zhou ; Yinfei Xu ; Wei Yu ; Jun Chen, “On the Optimal Fronthaul Compression and Decoding Strategies for Uplink Cloud Radio Access Networks,”IEEE Transactions on Information Theory, vol. 62, no. 12, Dec. 2016.
- A. Zaidi and I. E. Aguerri, “Tutorial: Fronthaul Compression for Cloud Radio Access Networks,” The Thirteenth Internationl Symposium on Wireless Communication Systems (ISWCS’16), Sep. 20-23, 2016, in Pozna#, Poland.
- O. Simeone, S.-H. Park, O. Sahin and S. Shamai (Shitz), “Frontal Compression for C-RAN,” Chapter 14 in Cloud Radio Access Networks: Principles, Technologies, and Applications, T. Q. S. Quek, M. Peng, O. Simeone, and W. Yu, Eds. Cambridge University Press, Feb. 2017.
- Z. Guizani and N. Harmdi, ”CRAN, H-CRAN, and F-RAN for 5G systems: Key capabilities and recent advances,” International Journal of Network Management, pp. 1-22, 2017.

Thank you!

PRINCETON
Electrical
Engineering

Department of Electrical Engineering Seminar

Speaker: **Dr. Shlomo Shamai (Shitz)**
Technion-Israel Institute of Technology
Title: **Cloud Radio Access Networks, Distributed Information Bottleneck, and more: A Unified Information Theoretic View**
Date: **Thursday, June 14, 2018**
Time: 4:00 pm
Room: E-Quad, B205
Host: **Prof. H. Vincent Poor**

Abstract: We consider transmission over a cloud radio access network (CRAN) focusing on the framework of oblivious processing at the relay nodes (radio units), i.e., the relays are not cognizant of the users' codebooks.
This approach is motivated by future wireless communications (5G and beyond) and the theoretical results connect to a variety of different information theoretic models and problems. First it is shown that relaying a-la Cover-El Gamal, i.e., compress-and-forward with joint decompression and decoding, which reflects 'noisy network coding,' is optimal. The penalty of obliviousness is also demonstrated to be at most a constant gap, when compared to cut-set bounds. Naturally, due to the oblivious (nomadic) constraint the CRAN problem intimately comments to Chief Executive Officer (CEO) source(s) coding under a logarithmic loss distortion measure. Furthermore, we identify and elaborate on some interesting connections with the distributed information bottleneck model for which we characterize optimal tradeoffs between rates (i.e., complexity) and information (i.e., accuracy) in the discrete and vector Gaussian frameworks. Further connections to 'information combining' and 'common reconstruction' are also pointed out. In the concluding outlook, some interesting problems are mentioned such as the characterization of the optimal input distributions under users' power limitations and rate-constrained compression at the relay nodes,
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Joint work with: I.E. Aguerri (Paris Research Center, Huawei France) A. Zaidi (Universite Paris-Est, Paris) and G. Caire (USC-LA and TUB, Berlin)
The research is supported by the European Union's Horizon 2020 Research And Innovation Programme: no. 694630.

Bio: Shlomo Shamai (Shitz) received the B.Sc., M.Sc., and Ph.D. degrees in electrical engineering from the Technion---Israel Institute of Technology, in 1975, 1981 and 1986 respectively.

During 1975-1985, he was with the Communications Research Labs, in the capacity of a Senior Research Engineer. Since 1986 he is with the Department of Electrical Engineering, Technion---Israel Institute of Technology, where he is now a Technion Distinguished Professor, and holds the William Fondiller Chair of Telecommunications. His research interests encompass a wide spectrum of topics in information theory and statistical communications.

Dr. Shamai (Shitz) is an IEEE Fellow, an URSI Fellow, a member of the Israeli Academy of Sciences and Humanities and a foreign member of the US National Academy of Engineering. He is the recipient of the 2011 Claude E. Shannon Award, the 2014 Rothschild Prize in Mathematics/Computer Sciences and Engineering and the 2017 IEEE Richard W. Hamming Medal.