Bounding the Number of Mass Points of the Capacity-Achieving Input for the Amplitude and Power Constrained Additive Gaussian Channel

Semih Yagilı†, Alex Dytsko*, H. Vincent Poor‡, Shlomo Shamai (Shitz)**

†-‡Department of Electrical Engineering, Princeton University, Princeton, N.J. USA, 08544.
**Department of Electrical Engineering, Israel Institute of Technology, Haifa, Israel, 3200003.
E-mail: syagi@princeton.edu, adytsko@princeton.edu, hpoor@princeton.edu, sshlomo@ee.technion.ac.il

Abstract
We study the real and complex Additive Gaussian Channels (AGCs) with input amplitude constraints. For the real AGC model, it is well known that the capacity-achieving input distribution is discrete with finitely many mass points. Similarly, for the complex AGC model, it is well known that the amplitude of the capacity-achieving input has a distribution that is discrete with finitely many mass points. However, due to the previous proof technique, neither the exact numbers of mass points of the optimal input distributions in these settings nor bounds on them were available. Here we provide an alternative proof of the discreteness of the capacity-achieving input distributions and produce the first firm upper bounds on the number of mass points, paving an alternative way for approaching many such problems. The key ingredients of this new proof technique are Karlin’s oscillation theorem and Tijden’s number of zeros lemma.

Introduction

Main Results

A Bound on Number of Mass Points of \(X_N\) for Problem 1

\[ \frac{1}{\sqrt{2\pi}} \leq \sup_{|a| \leq A} \left( P_{X_N}^* \right) \leq a_0 A^2 + a_1 A + a_0, \]

where

\[ a_0 = \frac{\pi}{2} \log(1 + 2A^2), \]

\[ a_1 = \frac{1}{2} \sqrt{2\pi}, \]

\[ a_0 = \pi \log(1 + 2A^2). \]

A Bound on Number of Mass Points of \(X_N\) for Problem 2

\[ \frac{1}{\sqrt{2\pi}} \leq \sup_{|a| \leq A} \left( P_{X_N}^* \right) \leq a_0 A^2 + a_1 A + a_0, \]

where

\[ a_0 = \frac{\pi}{2} \log(1 + 2A^2), \]

\[ a_1 = \frac{1}{2} \sqrt{2\pi}, \]

\[ a_0 = \pi \log(1 + 2A^2). \]

A Bound on Number of Mass Points of \(X_N\) for Problem 3

\[ \frac{1}{\sqrt{2\pi}} \leq \sup_{|a| \leq A} \left( P_{X_N}^* \right) \leq a_0 A^2 + a_1 A + a_0, \]

where

\[ a_0 = \frac{\pi}{2} \log(1 + 2A^2), \]

\[ a_1 = \frac{1}{2} \sqrt{2\pi}, \]

\[ a_0 = \pi \log(1 + 2A^2). \]

Remarks on the Main Results

• The lower bounds in Theorems 1 and 2 follow from the entropy-power inequality.
• For the upper bounds, we rely on Tijden’s Lemma on the number of zeros of an analytic function [3, Lemmas 1 and 2].
• The lower and upper bounds in Theorem 3 are order-tight.

Remarks on the Main Results

• The lower bounds in Theorems 1 and 2 follow from the entropy-power inequality.
• For the upper bounds, we rely on Tijden’s Lemma on the number of zeros of an analytic function [3, Lemmas 1 and 2]. Together, these two results find upper bounds for many similar problems.
• If \( P \geq A^2 \) as Theorem 2, we recover the result of Theorem 1.
• The lower and upper bounds in Theorem 3 are order-tight.

A Bound on Number of Mass Points of \(P_{X_N}^*\), for Problem 3

\[ \frac{1}{\sqrt{2\pi}} \leq \sup_{|a| \leq A} \left( P_{X_N}^* \right) \leq a_0 A^2 + a_1 A + a_0, \]

where

\[ a_0 = \frac{\pi}{2} \log(1 + 2A^2), \]

\[ a_1 = \frac{1}{2} \sqrt{2\pi}, \]

\[ a_0 = \pi \log(1 + 2A^2). \]

Relevant to the current technology, the problem of finding proper upper bounds on the number of mass points carries practical importance as much as its theoretical importance.

Main Components of Our Method

Tijden’s Number of Zeros Lemma

Definition The number of sign changes of a real-valued function \( f : \mathbb{R} \to \mathbb{R} \) is defined as

\[ \nu(f) = \sup_{x \in \mathbb{R}} \nu(f(x_0), \ldots, x_n), \]

where \( \nu(f(x_0), \ldots, x_n) \) is the number of sign changes of the sequence \( \{f(x_0), \ldots, x_n\} \).

Theorem 4 Let \( x(y) \) be a positive definite function and a pdf in \( x \) for every fixed \( y \). Assume that \( p \) is \( n \)-times differentiable with respect to \( x \) for arbitrary \( y \). Let \( x \) be a measure on the real line, and let \( \xi \) be a function with \( \nu(\xi(y)) = n \).

Karlin’s Oscillation Theorem

\[ \nu(\xi(y)) = n \]

is \( n \)-times differentiable with respect to \( x \), then either \( \nu(\xi) \leq n \) and \( \nu(\xi, \xi) \leq n \) is identically zero.

Other Related Problems

Problem 4. Gaussian Multiple Access Channel

\[ C(A_1, A_2, A_3) = \max_{X \in \mathcal{X}(A_1, A_2, A_3)} \int f(X_1, X_2, X_3) \]

\[ X_1 \in R \] \( \cap \) \( A_1 \)

\[ X_2 \in R \] \( \cap \) \( A_2 \)

\[ X_3 \in R \] \( \cap \) \( A_3 \)

\[ Z \sim N(0,1) \]

References


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