Bounding the Number of Mass Points of the Capacity-Achieving Input for the Amplitude and Power Constrained Additive Gaussian Channel

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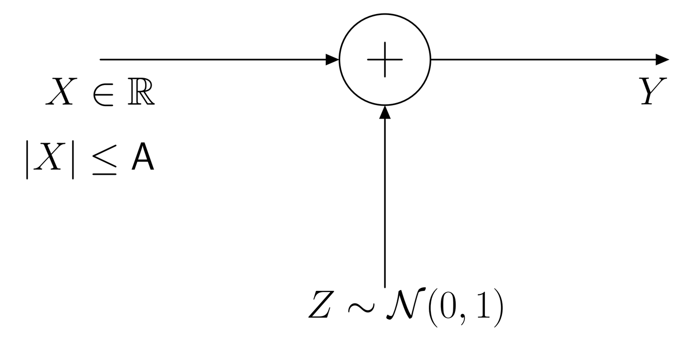
Abstract

We study the real and complex Additive Gaussian Channels (AGCs) with input amplitude constraints. For the real AGC model, it is well known that the capacity-achieving input distribution is discrete with finitely many mass points. Similarly, for the complex AGC model, it is well known that the amplitude of the capacity-achieving input has a distribution that is discrete with finitely many mass points. However, due to the previous proof technique, neither the exact numbers of mass points of the optimal input distributions in these settings nor bounds on them were available. We provide an alternative proof of the discreteness of the capacity-achieving input distributions and produce the first firm upper bounds on the number of mass points, paving an alternative way for approaching many such problems. The key ingredients of this new proof technique are Karlin's oscillation theorem and Tijdeman's number of zeros lemma.

Introduction

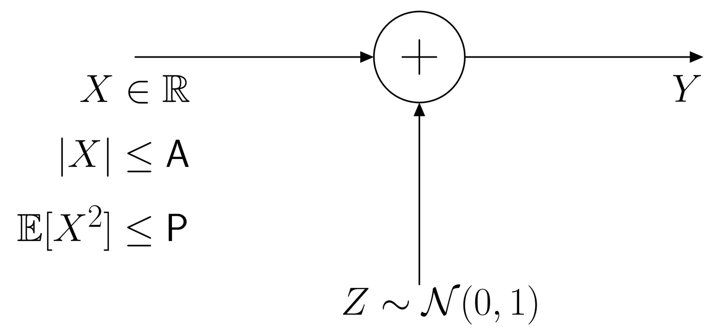
Problem 1. *AGC with Input Amplitude Constraint*

$$C(\mathsf{A}) = \max_{X\in\mathbb{R}\colon |X|\le \mathsf{A}} I(X;Y)$$



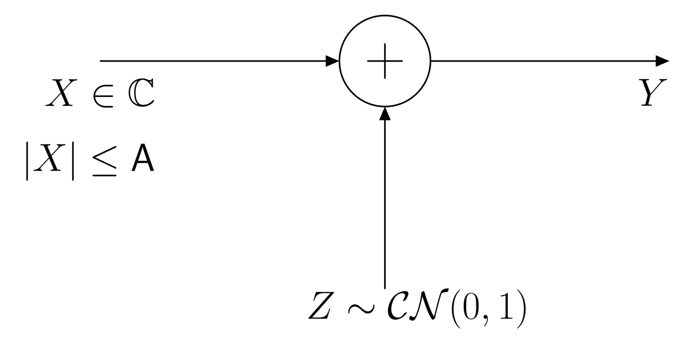
Problem 2. *AGC with Input Amplitude and Power Constraints*

$$C(\mathsf{A},\mathsf{P}) = \max_{\substack{X\in\mathbb{R}\colon |X|\le \mathsf{A}\\ \mathbb{E}[X^2]\le \mathsf{P}}} I(X;Y)$$



Problem 3. *Complex AGC with Input Amplitude Constraint*

$$C_{\mathsf{c}}(\mathsf{A}) = \max_{X\in\mathbb{C}\colon |X|\le \mathsf{A}} I(X;Y)$$



Theorem *[1, Corollaries 1 and 2]. In Problems 1 and 2, the capacity-achieving distribution $P_{X^\star}$ is* unique, symmetric, *and* discrete *with finitely many mass points.*

Theorem *[2, Theorem 1]. In Problem 3, the capacity-achieving distribution $P_{X^\star}$ is unique, radially symmetric, and the amplitude $|X^\star|$ of the capacity-achieving input $X^\star$ is discrete with finitely many mass points.*

Some Notes on Smith's and Shamai's Results:

- Both theorems have proofs that are based on a contradiction argument. Their proofs are not constructive, not even an explicit bound on the number of mass points is known.
- In the real AGC case, a bound on the number of mass points of the optimal input $X^\star$ as a function of amplitude constraint A is unknown except for certain particular cases of A. This is an open problem for almost *50 years!*
- Similarly, in the complex AGC case, a bound on the number of mass points of the amplitude of the optimal input, namely $|X^\star|$ as a function of the amplitude case A is unknown except for certain particular cases of A.
- Relevant to the current technology, the problem of finding proper upper bounds on the number of mass points carries practical importance as much as its theoretical importance.

Main Results

A Bound on Number of Mass Points of $P_{X^\star}$ for Problem 1

Theorem 1: AGC with Amplitude Constraint

For Problem 1, the optimal input $X^\star$ has a discrete distribution $P_{X^\star}$ where the number of mass points satisfy

$$\sqrt{1+\frac{2\mathsf{A}^2}{\pi \mathrm{e}}} \le |\mathrm{supp}(P_{X^\star})| \le a_2\mathsf{A}^2 + a_1\mathsf{A} + a_0,$$

where

$$a_2 = 9\mathrm{e} + 6\sqrt{\mathrm{e}} + 5,$$
$$a_1 = 6\mathrm{e} + 2\sqrt{\mathrm{e}},$$
$$a_0 = \mathrm{e} + 2\log\left(4\sqrt{\mathrm{e}}+2\right) + 1.$$

A Bound on Number of Mass Points of $P_{X^\star}$ for Problem 2

Theorem 2: AGC with Amplitude and Power Constraints

For Problem 2, the optimal input $X^\star$ has a discrete distribution $P_{X^\star}$ where the number of mass points satisfy

$$\sqrt{1+\frac{2\min\left\{\mathsf{A}^2, 3\mathsf{P}\right\}}{\pi \mathrm{e}}} \le |\mathrm{supp}(P_{X^\star})| \le a_{\mathsf{P}_2}\mathsf{A}_{\mathsf{P}}^2 + a_{\mathsf{P}_1}\mathsf{A}_{\mathsf{P}} + a_{\mathsf{P}_0},$$

where

$$\mathsf{A}_{\mathsf{P}} = \frac{\mathsf{AP}}{\mathsf{P} - \log(1+\mathsf{P})1\left\{\mathsf{P} < \mathsf{A}^2\right\}},$$
$$a_{\mathsf{P}_2} = (1+2\lambda_{\mathsf{P}})(9\mathrm{e}+6\sqrt{\mathrm{e}}+1) + 2(2-\lambda_{\mathsf{P}})(1-2\lambda_{\mathsf{P}}),$$
$$a_{\mathsf{P}_1} = (1+2\lambda_{\mathsf{P}})(6\mathrm{e}+2\sqrt{\mathrm{e}}),$$
$$a_{\mathsf{P}_0} = (1+2\lambda_{\mathsf{P}})\mathrm{e} + 2\log\left(\frac{2+4\sqrt{\mathrm{e}}(1+2\lambda_{\mathsf{P}})}{1-2\lambda_{\mathsf{P}}}\right) + 1,$$
$$\lambda_{\mathsf{P}} = \frac{\log(1+\mathsf{P})}{2\mathsf{P}} 1\left\{\mathsf{P} < \mathsf{A}^2\right\}.$$

A Bound on Number of Mass Points of $P_{|X^\star|}$ for Problem 3

Theorem 3: Complex AGC with Amplitude Constraint

For Problem 3, the amplitude of the optimal capacity-achieving random variable, namely $|X^\star|$, has a discrete distribution $P_{|X^\star|}$ where the number of mass points satisfy

$$\frac{1}{2\pi}\left(1+\frac{1}{2\mathrm{e}}\mathsf{A}^2\right) \le \left|\mathrm{supp}\left(P_{|X^\star|}\right)\right| \le a_{\mathsf{c}_2}\mathsf{A}^2 + a_{\mathsf{c}_1}\mathsf{A} + a_{\mathsf{c}_0},$$

where

$$a_{\mathsf{c}_2} = \left(2\mathrm{e} + \sqrt{3(2\mathrm{e}+1)} + \frac{5}{2}\right)(2+\sqrt{2}) + \frac{3}{2},$$
$$a_{\mathsf{c}_1} = 2\mathrm{e} + \sqrt{3(2\mathrm{e}+1)} + \frac{5}{2},$$
$$a_{\mathsf{c}_0} = 2\mathrm{e} + \sqrt{3(2\mathrm{e}+1)} + \frac{7}{2}.$$

Remarks on the Main Results

- The lower bounds in Theorems 1, 2 and 3 follow from the entropy-power inequality.
- For the upper bounds, we rely on Tijdeman's Lemma on the number of zeros of an analytic function [3, Lemma 1] and Karlin's Oscillation Theorem [4, Theorem 3]. Together, these two results find upper bounds for many similar problems.
- If $\mathsf{P} \ge \mathsf{A}^2$ in Theorem 2, we recover the result of Theorem 1.
- The lower and upper bounds in Theorem 3 are **order-tight**. That is, for the capacity-achieving input $X^\star$ in the complex AGC,
$$\left|\mathrm{supp}\left(P_{|X^\star|}\right)\right| = \Theta(\mathsf{A}^2).$$
- The order of the lower bounds in Theorems 1 and 2 are $\mathrm{O}(\mathsf{A})$ while the order of the upper bounds are $\mathrm{O}(\mathsf{A}^2)$. This is due to a limitation of the technique we use.
- The constants we found might be improved because those constants result from a suboptimal choice in a hard optimization problem.

Main Components of Our Method

Tijdeman's Number of Zeros Lemma

Lemma *[3, Lemma 1]. Let R, s, t be positive numbers such that $s > 1$. For the complex valued function $f \neq 0$ which is analytic on $|z| < (st+s+t)R$, its number of zeros $\mathrm{N}(\mathcal{D}_R, f)$ within the disk $\mathcal{D}_R = \{z\colon |z| \le R\}$ satisfies*

$$\mathrm{N}(\mathcal{D}_R, f) \le \frac{1}{\log s}\left(\log \max_{|z|\le(st+s+t)R} |f(z)| - \log \max_{|z|\le tR} |f(z)|\right).$$

Karlin's Oscillation Theorem

Definition *The number of sign changes of a real-valued function $\xi\colon \mathbb{R}\to\mathbb{R}$ is defined as*

$$\mathscr{S}(\xi) = \sup_{m\in\mathbb{N}} \sup_{y_1,\ldots,y_m} \mathscr{N}(\xi|y_1,\ldots,y_m),$$

where $\mathscr{N}(\xi|y_1,\ldots,y_m)$ is the number of sign changes of the sequence $\{\xi(y_i)\}_{i=1}^m$, for $y_i < y_{i+1}$.

Theorem *[4, Theorem 3]. Let $p(x,y)$ be a positive definite function and a pdf in x for every fixed y. Assume that p is n-times differentiable with respect to x for arbitrary y. Let ν be a measure on the real line, and let ξ be a function with $\mathscr{S}(\xi) = n$. If*

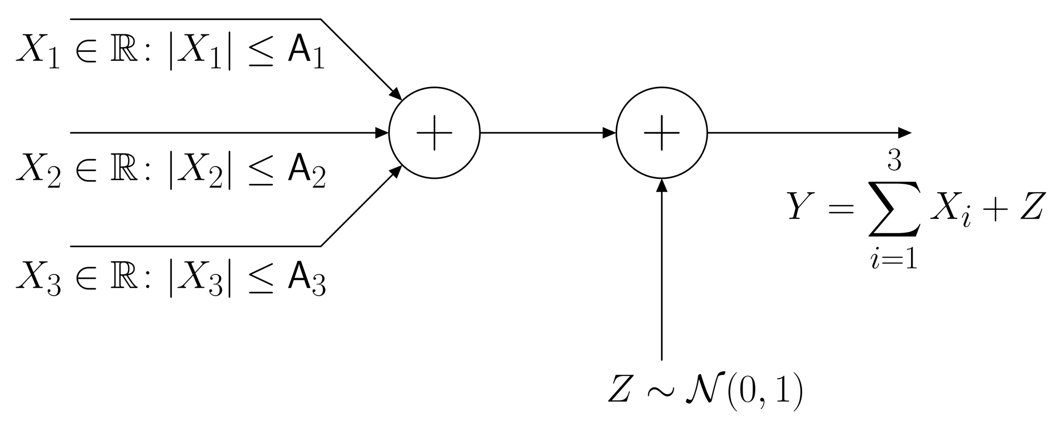
$$\Xi(x) = \int \xi(y)p(x,y)\mathrm{d}\nu(y),$$

is n-times differentiable with respect to x, then either $\mathscr{S}(\Xi) \le n$ and $\mathrm{N}(\mathbb{R},\Xi) \le n$, or Ξ is identically zero.

Other Related Problems

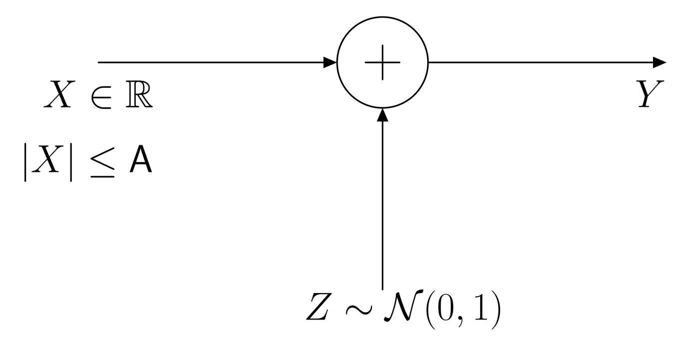
Problem 4. *Gaussian Multiple Access Channel*

$$C(\mathsf{A}_1,\mathsf{A}_2,\mathsf{A}_3) = \max_{X^3\in\mathbb{R}^3\colon X_i\le \mathsf{A}_i\ \forall i} I(X_1,X_2,X_3;Y)$$



Problem 5. *Estimation Theory–Least Favorable Prior*

$$\mathcal{M}(\mathsf{A}) = \max_{X\in\mathbb{R}\colon |X|\le \mathsf{A}} \mathrm{MMSE}(X|Y)$$



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