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The Filtered Gaussian Primitive Diamond Channel

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***Abstract*—We investigate the special case of diamond relay comprising a Gaussian channel with identical frequency response between the user and the relays and fronthaul links with limited rate from the relays to the destination. We use the oblivious compress and forward (CF) with distributed compression and decode and forward (DF) where each relay decodes the whole message and sends half of its bits to the destination. We derive the achievable rate by using optimal time-sharing between DF and CF, which is advantageous over superposition of CF and DF. The optimal time sharing proportion between DF and CF and power and rate allocations are different at each frequency and are fully determined.**

***Index Terms*—Diamond Relay Channel, Information Bottleneck, Compress and Forward, Decode and Forward, Distributed Compression**

I. Introduction

MODERN and future communication systems serving mobile users by a fixed infrastructure, either cellular systems or cell-less systems, are using distributed base-stations comprising a central unit (CU) performing signal processing and control, denoted as "Destination" in Fig. 1, and remote radio heads (RRH) performing the radio frequency (RF) functions of amplification, reception, antennas and basic signal processing, denoted "Relays" in Fig.1. The RRHs and the CU are connected by digital links which may be optical fiber links, digital RF links or internet based systems. This work examines the performance limits in the uplink from the mobile device transmitting RF signals, 'X' in Fig. 1, to the destination accounting for limited available communication rate at the fronthaul links, connecting the RRHs to the CU. This problem was examined in [1] where we used the oblivious CF with distributed compression. Here we expand the system model by allowing the relays to do time-sharing between CF and DF. As in [1] we extend the examination to account for frequency selective channels, assuming equal characteristics, from the mobile device to the RRHs. The optimal solution divides time and frequency assignments between CF and DF. In the optimal solution, DF must clearly comply exactly with classical water-filling, and CF must comply with the rules presented in [1]. For our system scheme the time-sharing between CF and DF is proved in [2] to be advantageous over superposition coding of CF and DF. System rate optimization using Lagrange multipliers and Karush–Kuhn–Tucker (KKT) conditions for various problems was also studied in [3]. However, it does not include the diamond relay system, which requires also constraints on the fronthaul links rates in addition to the applied power constraints.

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A. System model

The system model is a real Gaussian signal X over two additive white Gaussian noise (AWGN) relay channels. Each relay channel has signal to noise ratio (SNR) equals to P in the frequency-flat case. For the frequency selective case the channel response also affects the SNR as the signal X is multiplied by the frequency dependent filter value. Each relay has a rate limited encoder connected to the destination decoder via a fronthaul link. In this paper we limit the model to the case where $H_1(f) = H_2(f) = H(f)$. Each relay channel has limited bit rate C[bits/ channel use] fronthaul channel from the encoder to the decoder at the destination. The relay encoders, as pointed out, do time sharing between CF and DF, and they do not communicate with each other. We aim to maximize the mutual information $I(X;\hat{X})$ between the source X and the destination $\hat{X}$ subject to the source power constraint and a fronthaul rate constrained link between the relays and the destination. In practice, reliable communication rate approaching $I(X;\hat{X})$ will be achieved by standard encoding of X.

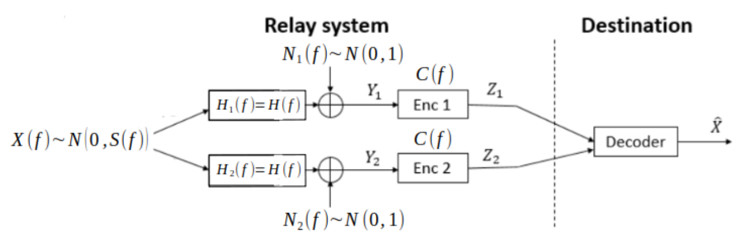

Figure 1: Gaussian diamond relay channel scheme for the frequency selective case

B. Information bottleneck

The Information Bottleneck (IB) method [4] can be used in order to find an optimal mapping according to a balance between maximizing mutual information of source and destination, while being constrained by the relay to destination rate. For the oblivious system with no interference, this method finds optimal solution. An extension for this method, named distributed bottleneck, was shown to be optimal for the oblivious case in [5]. For the non-oblivious case, DF can be used by having the relays decoding the messages, functioning as receivers in a broadcast channel. We showed the optimal rate of distributed CF over frequency-selective channels in [1] and the methodology is used in this paper.

II. Preliminaries

In this section we summarize previous results of discrete time frequency-flat Gaussian diamond relay channel. The transmitter uses classic codes, for real Gaussian-distributed X, and the channel to each relay is AWGN.

A. Upper bounds

For the general diamond relay channel, the cut-set upper bound is known as

$$\begin{aligned} &R \leq I(X;Y_1,Y_2) \\ &R \leq I(X;Y_1) + C_2 \\ &R \leq I(X;Y_2) + C_1 \\ &R \leq C_1 + C_2 \end{aligned} \tag{1}$$

In our Gaussian model with $SNR_1 = SNR_2 = P$ and $C_1 = C_2 = C$ the bound becomes

$$R_{cutset} = min\left(\frac{1}{2}log_2(1+2P), \frac{1}{2}log_2(1+P) + C, 2C\right) \tag{2}$$

In this paper we compare our scheme results to the cut-set upper bound at Eq. (2), and to a tighter upper bound shown in [6].

B. Compress and forward

In this transmission method, on each transmission block the relay quantizes its received message, encodes and transmits it to the destination in the following transmission block. In this method the relays are oblivious about the encoding scheme, which gives the system various advantages which are discussed in [1]. The system rate when using CF with joint decompression and decoding was shown in [7] and is based on distributed bottleneck as in [5]. Noisy network coding described in [8], was shown in [5] to be equivalent for the oblivious CF processing, along with the above method. We use Eq. (3) that was derived in [7] with our calculation. We define here the SNR of each AWGN relay channel as P_{CF} and the fronthaul channel rate limit as C_{CF}.

$$\begin{aligned} R_{CF} =& \frac{1}{2}log_2\left[1 + 2P_{CF}\cdot 2^{-4C_{CF}}\cdot\left(2^{4C_{CF}} + P_{CF}\right.\right. \\ &\left.\left. - \sqrt{P_{CF}^2 + (1+2P_{CF})\cdot 2^{4C_{CF}}}\right)\right] \end{aligned} \tag{3}$$

Our system model was investigated in [9], which derived a rate equation that equals the rate in Eq. (3).

C. Decode and forward

In this transmission method, each relay decodes its message and sends it to the destination through a fronthaul noiseless link. The DF rate for our system is the known Gaussian broadcast channel capacity with $SNR_1 = SNR_2 = P_{DF}$

$$R_{DF} = \frac{1}{2}log_2(1+P_{DF}) \tag{4}$$

and the required fronthaul channel rate is $C_{DF} \geq \frac{R_{DF}}{2} = \frac{1}{4}log_2(1+P_{DF})$ because each relay is required to send half of the message to the destination. Optimal frequency allocation solution is derived in [2].

III. Flat frequency response analysis

We now investigate the diamond relay system shown in Fig. 1 for the flat frequency response case. The transmitter uses classic codes, for real Gaussian-distributed X. In this case the channel response is set to be $H_1(f) = H_2(f) = 1$, so it does not affect X, while the channel to each relay is AWGN. Each relay uses time sharing between CF and DF. When using CF the compression in each relay is done by using remote source-coding with distributed compression. When using DF each relay decodes the message and transmit half of the bits to the destination.

A. System Rates

For the flat frequency response case, the cut-set upper bound of the diamond relay system is as in Eq. (2), the CF rate is as in Eq. (3) and DF rate is as in Eq. (4). Fig. 2 shows the above rates as a function of the relay to destination rate C_r and $P = 3$. It can be seen that for low relay rate the DF system rate is higher while for high relay rates the CF system rate is higher. We can immediately infer that using a simple switch between CF and DF and choosing the better one for a certain relay rate would have a better performance than using only one of them.

B. Time sharing

Now we will investigate the optimal solution of the time sharing scheme. In the first phase, both relays transmit DF over time T_{DF} with power P_{DF} and fronthaul rate C_{DF}. In the second phase both relays transmit CF over time T_{CF} with power P_{CF} and fronthaul rate C_{CF}. As seen in Fig. 2, an optimal solution of DF with a given P_{DF} will allocate the minimal required relay rate C_{DF} that achieves the DF rate at its time slot, because increasing it would not increase the system rate. As shown in section II, the minimal rate can be calculated directly from DF allocated power, therefore C_{DF} would be a function of P_{DF} and not a variable of the optimization problem. Now we will write the time sharing optimization problem for the flat frequency response case

$$\begin{aligned} &\max_{P_{DF},P_{CF},C_{CF},T_{CF},T_{DF}} \quad T_{DF}\cdot R_{DF} + T_{CF}\cdot R_{CF} \\ &s.t. \\ &0 \leq T_{DF} \leq 1 \\ &0 \leq T_{CF} \leq 1 \\ &0 \leq T_{DF} + T_{CF} \leq 1 \\ &0 \leq T_{DF}\cdot P_{DF} + T_{CF}\cdot P_{CF} \leq P \\ &0 \leq T_{DF}\cdot C_{DF} + T_{CF}\cdot C_{CF} \leq C \\ &C_{DF} = \frac{R_{DF}}{2} = \frac{1}{4}log_2(1+P_{DF}) \end{aligned} \tag{5}$$

The rates in Eq. (5) are those written in Eq. (3) and Eq. (4). The optimal solution for both DF and CF allocation is shown in [2] for the frequency-selective case. The flat frequency response model optimization can be solved by simpler two variable grid search, which we used in order to verify our proposed method results. The optimal solution as a function of the relay to destination rate C_r with power constraint $P = 3$ is shown in Fig. 2. Solution is obtained using the Lagrange multipliers method described in [1] and also in Section IV for the frequency-selective case. We use MATLAB Symbolic Toolbox in order to find the analytic solutions expressions and the Lagrange multipliers region, then we use Optimization Toolbox to find the optimal solution. It can be seen that at low relay rates the DF part is dominant. As the relay rate increases, the system rate increases and Tdf decreases, which indicate that the CF stage becomes relevant as it can use high relay rate and increase the total channel rate. This is until the optimal rate coincides with the CF rate. From this behavior we can infer that CF consumes more link rate resources than DF and therefore is used only when there is enough excess rate than using only DF. We applied the same method in

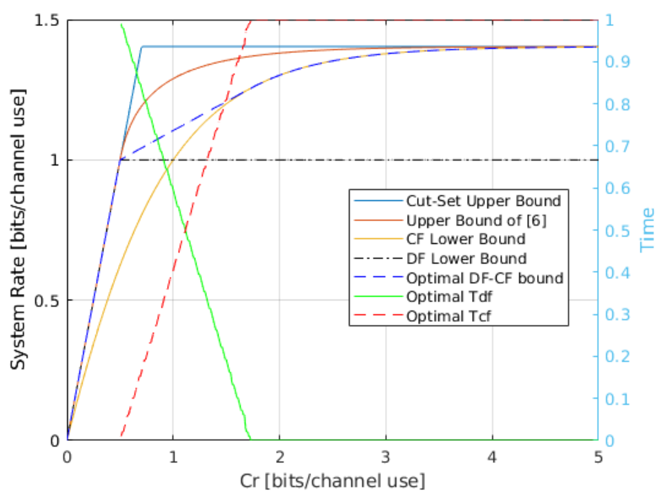

Figure 2: Gaussian diamond relay system upper bounds and CF, DF rates with and without time-sharing for various relay rates

order to find optimal solution as a function of the power constraint P with relay to destination rate $C_r = 1$, which is shown in Fig. 3. At very low power only CF is allocated and only for part of the total time. It is expected that CF will be preferred in this region as it is known, for example from [10], that for the single relay channel CF is preferred over DF in the case where the relays are far from the source, which can be thought as low power case in our system. As the power increases, DF is being allocated so we have time-sharing between CF and DF. T_{CF} increases until it achieves maximum value, then it decreases until reaching T_{DF} and after that it decreases to zero. From this behavior we can infer that DF consumes more power resources than CF and therefore is used only when there is enough power. From this figure we can infer that using time-sharing will improve the system rate and the largest improvement is in the case where the power value is at the medium values range, this is where the relays are at medium range from the source. When the relays are close to the source the time-sharing will prefer DF for best performance, and when they are very far from the source it will prefer CF. We note that for the frequency-flat case, instead of time-sharing, the same optimization can also be thought as allocating DF and CF in separate frequency bands using the same equations.

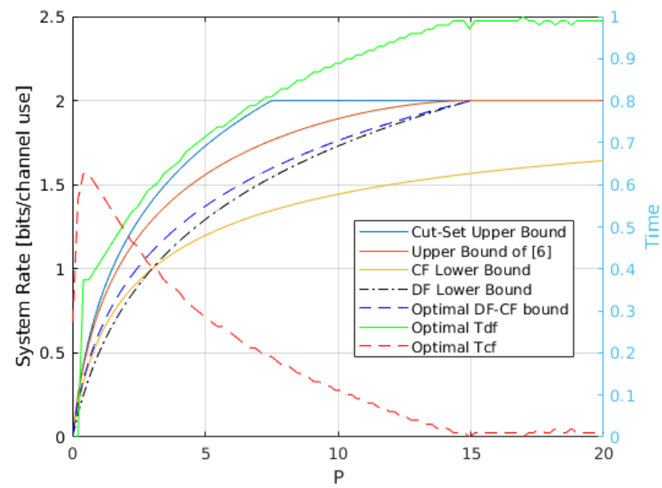

Figure 3: Gaussian diamond relay system upper bounds and CF, DF rates with and without time-sharing for various powers

IV. Frequency-selective case analysis

We now investigate the diamond relay system shown in Fig. 1 for the frequency selective case. In this case the channel to the relays comprises a frequency response set to $H_1(f) = H_2(f) = H(f)$ and an independent AWGN at each relay, as shown in Fig. 1. Each relay uses time sharing between CF and DF as described in Section III.

A. Generalized Water-pouring

In this paper we use the same generalized water-pouring approach that was explained and used in [1]. We derive DF solution in [2] and show that the additional time-sharing variables do not affect the optimal solution. This allows us to use previous results for CF, shown in [1].

B. System rates

We can now generalize the optimization problem of the frequency-selective case. The DF rate for this case using Eq. (4) is

$$R_{DF}(f) = \frac{1}{2} log_2 \left(1 + S_{DF}(f) \cdot |H(f)|^2\right) \tag{6}$$

and the required relay rate is $C_{DF}(f) \geq R_{DF}(f) = \frac{1}{2}log_2(1+ S_{DF}(f) \cdot |H(f)|^2)$. The CF rate for this case using Eq. (3) is

$$R_{CF}(f) = \frac{1}{2}log_2 \Big[1 + 2A(f) \cdot 2^{-2C_{CF}(f)} \cdot \Big(2^{2C_{CF}(f)} + A(f) \tag{7}$$

$$- \sqrt{A(f)^2 + (1 + 2A(f)) \cdot 2^{2C_{CF}(f)}}\Big)\Big]$$

$$A(f) \triangleq S_{CF}(f) \cdot |H(f)|^2$$

C. Time sharing

Now we can write the frequency dependent optimization problem

$$max \int_0^W [T_{DF}(f)\cdot R_{DF}(f)+T_{CF}(f)\cdot R_{CF}(f)]\cdot 2df \tag{8}$$

$$s.t.$$
$$0\le T_{DF}(f)\le 1$$
$$0\le T_{CF}(f)\le 1$$
$$0\le T_{DF}(f)+T_{CF}(f)\le 1$$
$$0\le \int_0^W [T_{DF}(f)\cdot S_{DF}(f)+T_{CF}(f)\cdot S_{CF}(f)]\cdot df \le P$$
$$0\le \int_0^W [T_{DF}(f)\cdot C_{DF}(f)+T_{CF}(f)\cdot C_{CF}(f)]\cdot df \le C$$
$$C_{DF}(f)=R_{DF}(f)=\frac{1}{2}log_2(1+S(f)\cdot|H(f)|^2)$$

Similar to [1] we now use the Lagrange multipliers method in order to solve this system. Since we choose the same CF solution as in [1], R_{CF} is concave with respect to S_{CF}, C_{CF}. R_{DF} is concave with respect to S_{DF} as the known logarithm function. Because R_{DF} does not depend on S_{CF}, C_{CF} and R_{CF} does not depend on S_{DF}, they both concave with respect to S_{DF}, S_{CF}, C_{CF}. Therefore linear combination of R_{CF} and R_{DF} is also concave with respect to S_{DF}, S_{CF}, C_{CF}. So our total system rate is concave and the optimization problem can be solved by the Lagrange multipliers method. The Lagrangian function is

$$\begin{aligned} L(f) =& 2\cdot[T_{DF}(f)\cdot R_{DF}(f)+T_{CF}(f)\cdot R_{CF}(f)] \\ &-\lambda_S\cdot[T_{DF}(f)\cdot S_{DF}(f)+T_{CF}(f)\cdot S_{CF}(f)] \\ &-\lambda_C\cdot[T_{DF}(f)\cdot C_{DF}(f)+T_{CF}(f)\cdot C_{CF}(f)] \end{aligned} \tag{9}$$

Because $C_{DF}(f)$ is a function of $S_{DF}(f)$ the Lagrangian function variables are $S_{DF}(f), S_{CF}(f), C_{CF}(f)$. We know that an optimal solution at each frequency must satisfy the following KKT conditions

$$\nabla L(f)=\left(\frac{dL}{dS_{DF}},\frac{dL}{dS_{CF}},\frac{dL}{dC_{CF}}\right)(f)=(0,0,0) \tag{10}$$
$$\lambda_S\left[\int[T_{DF}(f)\cdot S_{DF}(f)+T_{CF}(f)\cdot S_{CF}(f)]\cdot df-P\right]=0$$
$$\lambda_C\left[\int[T_{DF}(f)\cdot C_{DF}(f)+T_{CF}(f)\cdot C_{CF}(f)]\cdot df-C\right]=0$$

Using CF and DF Lagrange multipliers equations shown in [2] we can derive bounds on the Lagrange multipliers that give an outer bound for the required region - the region where both solutions are feasible in which we would get a time sharing solution. However, this outer bound is not necessarily a region with a time sharing solution to the problem, as it could also contain regions where only one solution, either CF or DF, is feasible. This is shown in Fig. 5. For the optimal pair of λ_S and λ_C the equations in [2] provide $S_{CF}, C_{CF}, S_{DF}, C_{DF}$, but not T_{CF} and T_{DF}. Using those powers and rates values we then optimize T_{CF} and T_{DF} by using linear programming (LP) methods. Their optimal solution must satisfy the total power and rate constraints. We describe the LP optimization problem and summarize the optimization procedure in [2].

D. Results

In this section we will show some results of optimal allocation. The frequency selective case optimization was done using Python Scipy. First we examine a channel response monotonically increasing with frequency. With bandwidth of W=10[Hz] and with power and rate constraints P=100 and C=9 the allocation result is shown in Fig. 4. As can be seen,

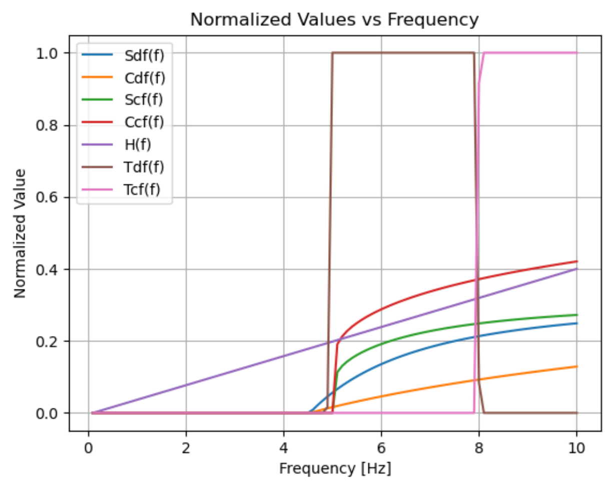

Figure 4: Monotonically increasing filter frequency allocation

the domain is devided into 3 regions. The first region with low filter values has no allocation. Second region has only DF allocation. Third region has only CF allocation. Between the regions there are two points of time sharing, first point does partly DF and second point does time sharing between CF and DF. This behavior corresponds to region 1 of Proposition 1. In [2] we calculate the total system rate achieved with optimal time sharing between CF and DF for the same frequency-selective filter used in [1]. The result is a rate of 7.5, compared to a lower value of 6.8 achieved with only CF in [1]. This is closer to the collaborative encoding upper bound rate of 8.16 also shown in [1]. Therefore time sharing of CF with DF reduces by half the gap between only CF scheme rate and the collaborative encoding upper bound rate.

V. Optimal solution properties

Next we analyze the behavior of the optimal solution for the frequency selective case.

Lemma 1: The solution for each Lagrange multipliers point divides the channel frequency bands into two types according to the filter value at each band and a filter value threshold H_{TH}.

1) Where $S_{DF} > S_{CF}$ for $H(f) < H_{TH}$.
2) Where $S_{CF} > S_{DF}$ for $H(f) > H_{TH}$.

Proof: The proof is shown in [2].

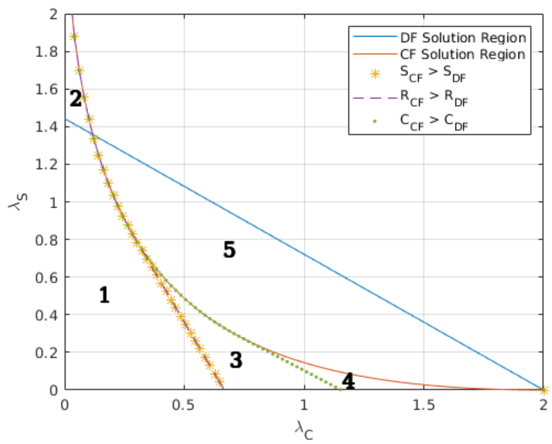

Figure 5: Regions of CF and DF solutions on the Lagrange multipliers grid. Below the lines is the region where $S_{CF} > S_{DF}$.

In Fig. 5 we show the regions border lines on the Lagrange multipliers grid. Below the CF and DF solution region lines are the regions where the solutions are feasible, where we define a non feasible solution if it is either negative or imaginary. Below the power, relay rate and channel rate lines is where CF has higher value. We now describe them according to the numbers shown on Fig. 5. Solution regions:

1) This region is where both CF and DF are allocated and CF has higher power, channel rate and relay rate.
2) In this region only CF can be allocated,
3) Here CF has higher relay rate.
4) DF has higher power, channel rate and relay rate.
5) Only DF is allocated.

We can see in Fig. 5 that CF consumes less relay rate on the region lines of power (yellow) and system rate (green), therefore the optimal solution would use CF there. In region 3 one can easily show that DF is preferable by assigning in its channel rate equation power of S_{CF} which is lower than S_{DF} there. This approach does not change the channel rate region, thus allowing us to compare the solutions by only the required relay rate. DF relay rate is smaller in this region, therefore the optimal solution would prefer it there. The lines equations are calculated in [2]. Next we refer to the case of two frequencies with optimal allocation, and let suppose that in each one of them there is CF and DF part. To generalize this we will divide each frequency into part A and part B, each could be either DF or CF. This is demonstrated in [2]. Theorem 1 shows the optimal solution behavior at this case.

Theorem 1: We define A and B to denote CF and DF respectively or in reverse order, that is, A may be CF and B denotes DF or A may be DF and then B denotes CF. We also define $\epsilon > 0$ as $H(f_2) = (1+\epsilon)H(f_1)$ and $K > (1+\epsilon)^2$ as $S_A(f_1) = K \cdot S_B(f_2)$.

Then for two frequency bands with different filter values such that $H_2 = H(f_2) > H(f_1) = H_1$ and $S_A(f_1) > S_B(f_2)$ and if there is some time in f_2 allocated to B, then A is not allocated in f_1 in the optimal solution.

Proof: The proof is shown in [2].

Using the above results we now state Proposition 1.

Proposition 1: Let the channel filter $H(f)$ be continuos in f. As was stated in Lemma 1, the optimal (λ_C, λ_S) point divide the filter values into 2 regions. The optimal allocation in those regions will be

1) Region of higher $H(f)$ values, where $S_{CF} > S_{DF}$ and CF will be allocated at the higher channel gains and DF at the lower ones. Also a band of lowest channel gains may be left unused.
2) Region of lower $H(f)$ where $S_{CF} < S_{DF}$ and only DF will be allocated. A band of lowest channel gains may be left unused.

Proof: The proof is shown in [2].

VI. Conclusions

In this work the band limited symmetric primitive diamond relay channel is considered, where a single user is connected to two non-cooperating relay nodes via symmetric bandlimited and filtered Gaussian channels, while the relay nodes are connected to the final-end receiver via ideal fronthaul links of given capacity. We consider and optimize achievable schemes that account for decode and forward (DF), and distributed compress and forward (CF), and compare the achievable rates to the cut-set bound and the upper bound from [6]. This method is shown in [2] to be advantageous over superposition of CF and DF. From the above discussion we can conclude that for frequency dependent filter with allocation for both CF and DF, higher filter values would prefer CF allocation and lower filter values would prefer DF allocation. Filter value between CF and DF allocation would have time sharing and low filter values would not have allocation. We show that using CF and DF time sharing we can increase the total system rate relative to using only one of them and compare the results to those obtained in [1].

References

[1] A. Katz, M. Peleg, and S. Shamai, “Gaussian diamond primitive relay with oblivious processing,” in *2019 IEEE International Conference on Microwaves, Antennas, Communications and Electronic Systems (COMCAS)*, 2019, pp. 1–6.
[2] A. Katz, M. Peleg, and S. Shamai, “The filtered gaussian primitive diamond channel,” 2021. [Online]. Available: https://arxiv.org/abs/2101.09564
[3] C. Xing, Y. Jing, S. Wang, S. Ma, and H. V. Poor, “New viewpoint and algorithms for water-filling solutions in wireless communications,” 2020.
[4] N. Tishby, F. C. Pereira, and W. Bialek, “The information bottleneck method,” in *Proc. of the 37-th Annual Allerton Conference on Communication, Control and Computing*, 1999, pp. 368–377. [Online]. Available: https://arxiv.org/abs/physics/0004057
[5] I. Estella Aguerri, A. Zaidi, G. Caire, and S. Shamai Shitz, “On the capacity of cloud radio access networks with oblivious relaying,” *IEEE Transactions on Information Theory*, vol. 65, no. 7, pp. 4575–4596, 2019.
[6] X. Wu, A. Ozgur, M. Peleg, and S. S. Shitz, “New upper bounds on the capacity of primitive diamond relay channels,” in *2019 IEEE Information Theory Workshop (ITW)*, 2019, pp. 1–5.
[7] A. Sanderovich, S. Shamai, Y. Steinberg, and G. Kramer, “Communication via decentralized processing,” *IEEE Transactions on Information Theory*, vol. 54, no. 7, pp. 3008–3023, 2008.
[8] S. H. Lim, Y. Kim, A. El Gamal, and S. Chung, “Noisy network coding,” *IEEE Transactions on Information Theory*, vol. 57, no. 5, pp. 3132–3152, 2011.
[9] S. Ganguly, S.-E. Hong, and Y.-H. Kim, “On the capacity regions of cloud radio access networks with limited orthogonal fronthaul,” *arXiv:1912.04483*, 10 Dec 2019.
[10] G. Kramer, I. Marić, and R. D. Yates, “Cooperative communications,” *Foundations and Trends in Networking*, vol. 1, no. 3–4, pp. 271–425, 2007. [Online]. Available: http://dx.doi.org/10.1561/1300000004